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Observer-Based Quadratic Guaranteed Cost Control for Linear Uncertain Systems with Control Gain Variation

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Abstract

This study proposes a method for designing observer-based quadratic guaranteed cost controllers for linear uncertain systems with control gain variations. In the proposed approach, an observer is designed, and then a feedback controller that ensures the upper bound on the given quadratic cost function is derived. This study shows that sufficient conditions for the existence of the observer-based quadratic guaranteed cost controller are given in terms of linear matrix inequalities. A sub-optimal quadratic guaranteed cost control strategy is also discussed. Finally, the effectiveness of the proposed controller is illustrated by a numerical example. The result shows that the proposed controller is more effective than conventional methods even if system uncertainties and control gain variations exist.

Keywords: polytopic uncertainty, quadratic guaranteed cost control, observer-based controller, control gain variation, linear matrix inequality (LMI)

1. Introduction

In the design of control systems for dynamical systems, it is necessary to derive a mathematical model for the controlled system. If the mathematical model represents the control system precisely, then the desired control system can be designed by various control design strategies. However, it is unavoidable that there are some gaps between an original controlled system and its mathematical model, and these gaps are known as "uncertainty." Therefore, robust controller design methods that can explicitly deal with uncertainties have been well studied. A large number of robust control strategies have been proposed [1-3]. Most conventional robust controllers have been designed by solving linear matrix inequalities (LMIs) and have fixed gains that are designed by considering the worst-case variation for uncertainties.

In fact, it is desirable to design robust control systems with not only robust stability but also satisfactory control performance. To achieve this, Chang and Peng [4] proposed guaranteed cost control. In this design method, there is an upper bound on a given performance index. The degradation of the system performance caused by uncertainties is guaranteed to be below this bound. Many studies have adopted this concept [5-7]. In the work of Moheimani and Peterson [6], the Riccati equation approach [5] was extended to uncertain time-delay systems, and a guaranteed cost controller design method that solves a certain parameter-dependent Riccati equation was proposed. Yu and Chu [7] proposed a design method for guaranteed cost controllers for linear uncertain time-delay systems that uses the LMI approach.

Studies on robust control generally assume that the full state of the controlled systems can be measured. However, in practice, the full state of systems cannot be obtained due to practical constraints. To overcome this problem, some observer-based quadratic stabilizing controllers have been presented [8-11]. For example, Oya and Hagino [10] proposed an observer-based guaranteed cost controller for polytopic uncertain systems. The polytopic representation allows the structure of uncertainties to be directly represented.

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On the other hand, Keel and Bhattacharyya [12] pointed out that it is necessary for a controller to tolerate some uncertainty when the control input is implemented. Controller implementation involves the uncertainties inherent in analog-to-digital and digital-to-analog conversion and roundoff errors in numerical computations. Thus, a nonzero margin of tolerance is required for the controller design. Many design methods that consider control gain variation have been proposed [13-16]. Yang et al. [13] proposed a design method for H_∞ control for linear systems with addition control gain variation. Famularo et al. [14] considered not only control gain variations but also the uncertainty of the system matrix. Oya et al. [15] proposed a design method for a robust controller for linear uncertain systems with control gain perturbation. However, the problem of observer-based quadratic guaranteed cost control for linear uncertain systems with control gain variation has not been discussed.

This study proposes a method for designing an observer-based guaranteed cost controller for linear uncertain systems with control gain variation. In this study, the design approach is separated into two steps [10, 17]. In the first step, an observer is designed; in the second step, a feedback controller that guarantees the upper bound on the given quadratic cost function is derived. Sufficient conditions for the existence of the proposed controller are given in terms of LMIs. The proposed control system can thus be designed using software such as MATLAB's LMI Control Toolbox and Scilab's LMITOOL.

2. Preliminaries

This section presents two lemmas that are used in this study. Lemma 1 shows the relation between matrices and a positive constant. Lemma 2 is the Schur complement formula.

Lemma 1 [16]: For matrices P and H that have appropriate dimensions, the following formula is obtained.

$$PH + H^T P^T \le \gamma P P^T + \frac{1}{\gamma} H^T H \tag{1}$$

where γ is a positive constant.

Lemma 2 (Schur complement formula [16]): For a given constant real symmetric matrix Ξ, the following items are equivalent.

$$\text{(i) } \Xi = \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{pmatrix} > 0 \tag{2}$$

$$\text{(ii) } \Xi_{11} > 0 \text{ and } \Xi_{22} - \Xi_{12}^T \Xi_{11}^{-1} \Xi_{12} > 0 \tag{3}$$

$$\text{(iii) } \Xi_{22} > 0 \text{ and } \Xi_{11} - \Xi_{12} \Xi_{22}^{-1} \Xi_{12}^T > 0 \tag{4}$$

3. Problem Formulation

This study considers the linear uncertain system described by the following:

$$\dot{x}(t) = A(\theta) x(t) + B u(t) \tag{5}$$

$$y(t) = C x(t) \tag{6}$$

where $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{l \times n}$ denote the known nominal matrices, and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$ are the vectors of the state, control input, and output, respectively. The full state cannot be measured. In Eq. (5), $A(\theta)$ is supposed to have appropriate dimensions and the following structure:

$$A(\theta) = A + \sum_{k=1}^{N} \theta_k A_k \tag{7}$$

In Eq. (7), the matrix $A \in \mathbb{R}^{n \times n}$ represents the known nominal value for system parameters, and the matrix A_k, $k = 1, 2, \ldots\ldots$, N, denotes the structure of the uncertainties. The parameter $\theta = (\theta_1, \ldots\ldots, \theta_N)^T$ represents unknown parameters that belong to the following parameter set:

$$\Delta \triangleq \left\{ \theta \in \mathbb{R}^N \mid \sum_{k=1}^{N} \theta_k = 1, \theta_k \geq 0 \text{ for } k = 1, \ldots, N \right\} \tag{8}$$

Furthermore, for $\forall\theta \in \Delta$, it is assumed that the pair $(A(\theta), B)$ and $(C, A(\theta))$ are controllable and observable respectively. Now the following full-state observer is introduced:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + G(t)(y(t) - C\hat{x}(t)) \tag{9}$$

where $G(t) \in \mathbb{R}^{n \times l}$ is the observer gain matrix which is described as:

$$G(t) = G + \Delta G(t) \tag{10}$$

$$\left\| \Delta G(t) \right\| \leq \varepsilon_G \tag{11}$$

In Eqs. (10) and (11), $G(t) \in \mathbb{R}^{n \times l}$ shows the uncertainty for the observer gain matrix, and ε_G is a known constant that represents the upper bound for $\Delta G(t)$. In other words, $\Delta G(t)$ fluctuates in the range given in Eq. (11). The actual control input $u(t)$ is defined as:

$$u(t) \triangleq -K(t)\hat{x}(t) \tag{12}$$

where $K(t) \in \mathbb{R}^{m \times n}$ is the control gain matrix which satisfies the following relation:

$$K(t) = K + \Delta K(t) \tag{13}$$

$$\left\| \Delta K(t) \right\| \leq \varepsilon_K \tag{14}$$

In Eq. (14), $K(t) \in \mathbb{R}^{m \times n}$ represents the uncertainty of the controller gain matrix, and the known constant ε_K is the upper bound for $\Delta K(t)$. In this study, the control input with $\Delta K(t)$ is considered so as to design the observer-based quadratic guaranteed cost controller under control gain variation. Namely, the manipulated control input for the uncertain linear system in Eqs. (5) and (6) is $u(t) \triangleq -K\hat{x}(t)$. Fig. 1 shows the configuration of the proposed control system.

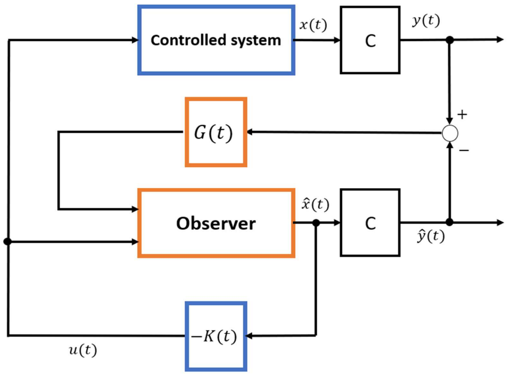

Fig. 1 Configuration of the proposed control system

For the linear uncertain system given in Eqs. (5) and (6), the following quadratic cost function is defined:

$$J = \int_0^\infty \left\{ x^T(t)Qx(t) + u^T(t)Ru(t) \right\} dt \tag{15}$$

where the weighting matrices $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ are positive definite and can be selected by designers. By introducing an estimation error vector $e(t) \triangleq x(t) - \hat{x}(t)$, from Eqs. (5) and (9), the following estimation error system is obtained:

$$\dot{e}(t) = (A(\theta) - G(t)C)e(t) + A_e(\theta)\hat{x}(t) \tag{16}$$

where $A_e(\theta)$ is the matrix given by $A_e(\theta) \triangleq A(\theta) - A$. Moreover, an augmented vector $x_e(t) \triangleq (\hat{x}(t)\ e(t))^T$ is introduced. Then, from Eqs. (5), (6), (9), (13), and (16), the following augmented system is derived:

$$\dot{x}_e(t) = \Omega(\theta)x_e(t) \tag{17}$$

$$\Omega(\theta) = \begin{pmatrix} A - BK(t) & G(t)C \\ A_e(\theta) & A(\theta) - G(t)C \end{pmatrix} \tag{18}$$

Moreover, by using the estimated error vector $e(t)$, the control input in Eq. (12), and the augmented vector $x_e(t)$, the quadratic cost function in Eq. (15) can be rewritten as follows:

$$J_{x_e} = \int_0^\infty x_e^T(t)\Gamma(t)x_e(t)dt \tag{19}$$

$$\Gamma(t) = \begin{pmatrix} Q + K^T(t)RK(t) & Q \\ Q & Q \end{pmatrix} \tag{20}$$

Note that because the weighting matrices $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ in Eq. (18) are positive definite, the matrix $\Gamma(t)$ in Eq. (20) is semi-positive definite. Applying Lemma 2 to Eq. (20), the semi-positive definiteness of the matrix $\Gamma(t)$ can be obtained as follows:

$$Q + K^T(t)RK(t) - QQ^{-1}Q = K^T(t)RK(t) \tag{21}$$

The definition of an observer-based quadratic guaranteed cost control is as follows.

Definition: The control input in Eq. (12) is an observer-based quadratic guaranteed cost control for the linear uncertain system in Eqs. (5) and (6) and the quadratic cost function in Eq. (19) provided that the closed-loop system in Eq. (17) is asymptotically stable for $\forall\theta \in \Delta$ and there exists a positive constant $\mathcal{J}^*(x_e(0))$ that satisfies $J_{x_e} \leq \mathcal{J}^*(x_e(0))$.

From the above discussion, the objective of this study is to design the observer gain matrix and the control gain matrix that guarantee the upper bound on the quadratic cost function in Eq. (19).

4. Main Results

This section shows an LMI-based design method for the observer gain matrix $G \in \mathbb{R}^{n \times l}$ and the control gain matrix $K \in \mathbb{R}^{m \times n}$ that ensures the upper bound of the quadratic cost function. It is difficult to design both gain matrices simultaneously because of the uncertainty parameters. Thus, the observer gain matrix G is first designed, and then the control gain matrix K is determined.

4.1. Design of observer gain matrix

This study considers the design of the observer gain matrix G that stabilizes the following system obtained by ignoring the estimate $\hat{x}(t)$ in Eq. (16):

$$\dot{\bar{e}}(t) = (A(\theta) - G(t)C)\bar{e}(t) \tag{22}$$

Now, let $\mathcal{V}_G(\bar{e}) \triangleq \bar{e}^T(t)Y_e\bar{e}(t)$ be a Lyapunov function candidate. $Y_e \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. From Lemma 1, Lemma 2, and a previously reported result [1], a sufficient condition for the asymptotical stability of the system in Eq. (22) is obtained as follows:

$$\begin{pmatrix} Y_e A(\theta) + A^T(\theta)Y_e - H_e C - C^T H_e^T + \gamma C^T C & \varepsilon_G Y_e \\ \varepsilon_G Y_e & -\gamma I_n \end{pmatrix} < 0, \quad \forall\, \theta \in \Delta_{\mathrm{vex}} \tag{23}$$

From Eq. (23), the observer gain matrix G can be easily designed as follows:

$$G = Y_e^{-1} H_e \tag{24}$$

4.2. Design of control gain matrix

In this section, the control gain matrix K that minimizes the upper bound on the quadratic cost function in Eq. (19) is designed. The following quadratic function is introduced as a Lyapunov function candidate:

$$\mathcal{V}_K(x_e) \triangleq x_e^T(t)\Lambda x_e(t) \tag{25}$$

where $\Lambda \in \mathbb{R}^{2n\times 2n}$ is a symmetric positive definite matrix. The time derivative of the quadratic function $\mathcal{V}_K(x_e)$ along the trajectory of the augmented system in Eq. (17) can be computed as:

$$\dot{\mathcal{V}}_K(x_e) = x_e^T(t)(\Omega^T(\theta)\Lambda + \Lambda\Omega(\theta))x_e(t) \tag{26}$$

Because the matrix $\Gamma(t)$ in Eq. (20) is semi positive definite, the following inequality is considered:

$$\Omega^T(\theta)\Lambda + \Lambda\Omega(\theta) + \Gamma(t) < 0, \quad \forall\, \theta \in \Delta_{\mathrm{vex}} \tag{27}$$

If a symmetric positive definite matrix Λ and a control gain matrix K that satisfy the matrix inequality in Eq. (27) exist, then the following relation holds:

$$\dot{\mathcal{V}}_K(x_e) < -x_e^T(t)\Gamma(t)x_e(t) < 0 \tag{28}$$

Namely, the augmented system in Eq. (17) is quadratically stable and $x_e(\infty) = 0$ holds. From $e(t) \triangleq x(t) - \hat{x}(t)$, the asymptotical stability of the linear uncertain system in Eqs. (5) and (6) is ensured. Furthermore, by integrating both sides of the inequality in Eq. (27) from 0 to ∞, the following equation can be obtained:

$$J_{x_e} = \int_0^\infty x_e^T(t)\Lambda x_e(t)dt < x_e^T(0)\Lambda x_e(0) = \mathcal{J}^*(x_e(0)) \tag{29}$$

Therefore, if matrices Λ and K that satisfy the LMI in Eq. (27) exist, the asymptotical stability of the system in Eqs. (5) and (6) is ensured and the upper bound on the quadratic cost function in Eq. (19) is given by Eq. (29).

Now, by introducing the auxiliary parameter $\delta \in \mathbb{R}^1$, the following matrix is considered (Remark 1):

$$\Gamma_\delta(t) = \begin{pmatrix} Q + \delta I_n + K^T(t)RK(t) & Q \\ Q & Q \end{pmatrix} \tag{30}$$

From Eqs. (20) and (30), the relation $\Gamma_\delta(t) - \Gamma(t) \geq 0$ holds. Therefore, the inequality in Eq. (27) also holds provided that the following condition is satisfied:

$$\Omega^T(\theta)\Lambda + \Lambda\Omega(\theta) + \Gamma_\delta(t) < 0, \quad \forall \theta \in \Delta_{\text{vex}} \tag{31}$$

Here, $\mathcal{S} \triangleq \text{diag}(S, S_e) \triangleq \Lambda^{-1}(S, S_e > 0 \in \mathbb{R}^{n\times n})$ and $\mathcal{W} \triangleq KS$ are defined. Then, pre- and post-multiplying Eq. (31) by $\mathcal{S}$, the condition in Eq. (31) can be written as:

$$\mathcal{S}\Omega^T(\theta) + \Omega(\theta)\mathcal{S} + \mathcal{S}\Gamma_\delta(t)\mathcal{S} < 0, \quad \forall \theta \in \Delta_{\text{vex}} \tag{32}$$

The inequality in Eq. (32) is organized as follows:

$$\begin{pmatrix} \Psi_{11}(t) & \Psi_{12}(t,\theta) \\ \Psi_{12}^T(t,\theta) & \Psi_{22}(t,\theta) \end{pmatrix} + \begin{pmatrix} S & 0 \\ 0 & S_e \end{pmatrix}\begin{pmatrix} Q + \delta I_n & Q \\ Q & Q \end{pmatrix}\begin{pmatrix} S & 0 \\ 0 & S_e \end{pmatrix}^T < 0, \quad \forall \theta \in \Delta_{\text{vex}} \tag{33}$$

$$\Psi_{11}(t) = AS + SA^T - B\mathcal{W} - \mathcal{W}^T B^T - S\Delta K^T(t)B^T - B\Delta K(t)S + SK^T(t)RK(t)S \tag{34}$$

$$\Psi_{12}(t,\theta) = SA_e^T(\theta) + GCS_e + \Delta G(t)CS_e \tag{35}$$

$$\Psi_{22}(t,\theta) = A(\theta)S_e + S_e A^T(\theta) - GCS_e - S_e C^T G^T - S_e C^T \Delta G^T(t) - \Delta G(t)CS_e \tag{36}$$

Using Lemma 1 and Lemma 2, the inequality of Eq. (33) can be described as follows:

$$\begin{pmatrix} \Psi_{11} & \Psi_{12}(\theta) & \mathcal{W}^T \\ \Psi_{12}^T(\theta) & \Psi_{22}(\theta) & 0 \\ \mathcal{W} & 0 & -R^{-1} + \xi\varepsilon_K^2 I_m \end{pmatrix} + \begin{pmatrix} S & 0 \\ 0 & S_e \end{pmatrix}\Gamma_\delta^*\begin{pmatrix} S & 0 \\ 0 & S_e \end{pmatrix}^T < 0, \quad \forall \theta \in \Delta_{\text{vex}} \tag{37}$$

$$\Psi_{11} = AS + SA^T - B\mathcal{W} - \mathcal{W}^T B^T + \alpha BB^T + \eta\varepsilon_G^2 I_n + \frac{\varepsilon_K^2}{\alpha}SS + \frac{1}{\xi}SS \tag{38}$$

$$\Psi_{12}(\theta) = SA_e^T(\theta) + GCS_e \tag{39}$$

$$\begin{aligned}\Psi_{22}(\theta) = {} & A(\theta)S_e + S_e A^T(\theta) - GCS_e - S_e C^T G^T + \beta\varepsilon_G^2 I_n + \frac{1}{\beta}S_e C^T CS_e \\ & + \frac{1}{\eta}S_e C^T CS_e\end{aligned} \tag{40}$$

Note that Γ_δ^* in Eq. (37) is the matrix expressed as:

$$\Gamma_\delta^* \triangleq \begin{pmatrix} Q + \delta I_n & Q \\ Q & Q \end{pmatrix} \tag{41}$$

One can easily see that the matrix Γ_δ^* is positive definite because its positive definiteness is equivalent to $Q+\delta I_n - QQ^{-1}Q = \delta I_n > 0$. Because the matrix Γ_δ^* is a positive definite, Lemma 2 can be applied to the inequality condition in Eq. (37). As a result, the following condition is obtained:

$$\Psi(\theta)=\begin{pmatrix} \Psi_{11} & \Psi_{12}(\theta) & \mathcal{W}^T & \varepsilon_K S & S & 0 & 0 & S & 0 \\ \Psi_{12}^T(\theta) & \Psi_{22}(\theta) & 0 & 0 & 0 & S_e C^T & S_e C^T & 0 & S_e \\ \mathcal{W} & 0 & -R^{-1}+\xi\varepsilon_K^2 I_m & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_K S & 0 & 0 & -\alpha I_n & 0 & 0 & 0 & 0 & 0 \\ S & 0 & 0 & 0 & -\xi I_n & 0 & 0 & 0 & 0 \\ 0 & CS_e & 0 & 0 & 0 & -\beta I_l & 0 & 0 & 0 \\ 0 & CS_e & 0 & 0 & 0 & 0 & -\eta I_l & 0 & 0 \\ S & 0 & 0 & 0 & 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$-\left(\Gamma_\delta^*\right)^{-1}$}} \\ 0 & S_e & 0 & 0 & 0 & 0 & 0 & & \end{pmatrix}<0, \quad \forall\theta\in\Delta_{\text{vex}} \tag{42}$$

$$\Psi_{11} = AS + SA^T - B\mathcal{W} - \mathcal{W}^T B^T + \alpha BB^T + \eta\varepsilon_G^2 I_n \tag{43}$$

$$\Psi_{12}(\theta) = SA_e^T(\theta) + GCS_e \tag{44}$$

$$\Psi_{22}(\theta) = A(\theta)S_e + S_e A^T(\theta) - GCS_e - S_e C^T G^T + \beta\varepsilon_G^2 I_n \tag{45}$$

The condition in Eq. (42) is an LMI for $\mathcal{S}>0, \mathcal{W}, \alpha>0, \beta>0, \eta>0$ and $\xi>0$. If the solution $\mathcal{S}>0, \mathcal{W}, \alpha>0, \beta>0, \eta>0$ and $\xi>0$ of the LMI in Eq. (42) exists, an observer-based quadratic guaranteed cost control law is obtained as follows:

$$u(t) \triangleq -K\hat{x}(t) \tag{46}$$

$$K = \mathcal{W}S^{-1} \tag{47}$$

From the above, the following theorem for designing the observer-based quadratic guaranteed cost controller is obtained:

Theorem 1: By solving the LMI in Eq. (23), the observer gain matrix G is derived as $G = Y_e^{-1}H_e$ in advance. If there exists the solution $\mathcal{S}>0, \mathcal{W}, \alpha>0, \beta>0, \eta>0$ and $\xi>0$ for $\exists\delta>0$ satisfying the LMI,

$$\Psi(\theta)<0, \quad \forall\theta\in\Delta_{\text{vex}} \tag{48}$$

then the control gain matrix K can be computed as $K=\mathcal{W}S^{-1}$ and the control law $u(t)=-K\hat{x}(t)$ becomes an observer-based quadratic guaranteed cost control.

4.3. Sub-optimal guaranteed cost control

Because the LMI in Eq. (42) has a convex solution $\mathcal{S}>0, \mathcal{W}, \alpha>0, \beta>0, \eta>0$ and $\xi>0$, it can be optimized by using software such as MATLAB's Robust Control Toolbox. In this section, the following optimization problem is considered:

$$\underset{\mathcal{S},\mathcal{W},\alpha,\beta,\eta,\xi}{\text{Minimize}}\ \mathcal{J}^*(x_e(0)) \text{ subject to Eq. (42) and } \mathcal{S}>0, \alpha>0, \beta>0, \eta>0, \xi>0 \tag{49}$$

If the optimization problem in Eq. (49) is solved, then a sub-optimal observer-based quadratic guaranteed cost control can be obtained. However, the upper bound $\mathcal{J}^*(x_e(0))$ in Eq. (49) depends on the initial augmented vector $x_e(0)$. Note that the error $e(0)$ cannot be utilized because the initial state $x(0)$ cannot be completely observed. Thus, to avoid this dependence, it is assumed that the initial vector $x_e(0)$ is a random vector that satisfies $\mathrm{E}[x_e(0)x_e^T(0)] = I_{2n}$ and $\mathrm{E}[x_e(0)] = 0$. Then, the upper bound on the quadratic cost function in Eq. (29) is given as $E\left[\mathcal{J}^*\left(x_e(0)\right)\right] = Tr\{\Lambda\}$. Therefore, the minimization problem of $Tr\{\Lambda\}$ minimized subject to the LMI constraint in Eq. (42) can be derived. Moreover, by introducing a complementary variable $\Sigma \in \mathbb{R}^{2n\times 2n}$, which is a symmetric positive definite matrix, the following relation is considered:

$$\Sigma \geq \Lambda > 0 \Leftrightarrow \begin{pmatrix} \Sigma & I_{2n} \\ I_{2n} & \mathcal{S} \end{pmatrix} \geq 0 \tag{50}$$

Then, the minimization problem of $Tr\{\Lambda\}$ can be transformed into that of $Tr\{\Sigma\}$ because the condition in Eq. (50) is an LMI in Σ and $\mathcal{S}$. Consequently, the optimization problem in Eq. (49) can be reduced to the following constrained convex optimization problem:

$$\underset{\Sigma,\mathcal{S},\mathcal{W},\alpha,\beta,\eta,\xi}{\text{Minimize}}\ Tr\{\Sigma\} \text{ subject to Eqs. (42) and (50) and } \mathcal{S} > 0, \alpha > 0, \beta > 0, \eta > 0, \xi > 0 \tag{51}$$

Finally, the following theorem can be obtained:

Theorem 2: If there exists the solution $\mathcal{S} > 0, \mathcal{W}, \alpha > 0, \beta > 0, \eta > 0$ and $\xi > 0$ that satisfies the constrained convex optimization problem in Eq. (51), there exists a sub-optimal observer-based quadratic guaranteed cost control. Note that by using the solution of the LMI in Eq. (23), the observer gain matrix G is derived as $G = Y_e^{-1}H_e$ in advance. If the solution to the constrained convex optimization problem is obtained, the control gain matrix K can be computed as $K = \mathcal{W}\mathcal{S}^{-1}$. Therefore, the control law $u(t) = -K\hat{x}(t)$ is a sub-optimal observer-based guaranteed cost control.

Remark 1: This study introduced the auxiliary parameter δ in Eq. (30). If δ is zero, the positive definiteness of the matrix Γ_δ^* is reduced to the relation $Q - QQ^{-1}Q = 0$. Then, Lemma 2 cannot be applied to the inequality in Eq. (37). If the parameter δ is a positive scalar, then the LMI in Eq. (42) can be obtained. However, if the parameter δ is set to a larger value, the result will be more conservative. Therefore, δ is set to be small as possible.

5. Simulation

This section demonstrates the effectiveness of the proposed method. As an example, the following aircraft model is considered [18]:

$$\dot{x}(t) = \begin{pmatrix} -0.091 & 0.097 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -5.43 & 0 & -0.686 & 3.62 & 2.87 & 0.638 \\ 0.56+\omega & 0 & 0 & -0.122 & 0.127 & 0.459 \\ 0 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & -10 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 10 & 0 \\ 0 & 10 \end{pmatrix} u(t) \tag{52}$$

$$y(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} x(t) \tag{53}$$

where the parameter ω represents the uncertainties and is assumed to vary in the range of $[-1.01.0]$. Let Case 1 and Case 2 be $\omega = -1.0$ and $\omega = 1.0$, respectively; these two cases are the worst cases of the uncertainties. It is assumed that the initial state and the initial estimate are $x(0) = (1.0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T$ and $\hat{x}(0) = (0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T$, respectively. The state variables are shown in Table 1. This simulation sets the weighting matrix of the quadratic cost function, the parameter δ, and the variations of K and G as follows:

$$Q = 1.0I_6,\ R = 4.0I_2,\ \delta = 1.0\times10^{-5} \tag{54}$$

$$\Delta K(t) = 0.1(1.0 - e^{-t}\,|\cos(10\pi t)|)\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{55}$$

$$\Delta G(t) = 0.1(1.0 - e^{-t}\,|\cos(10\pi t)|)(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1)^T \tag{56}$$

In addition, ε_K and ε_G are set as $\varepsilon_K = 0.35$ and $\varepsilon_G = 0.25$, respectively. By solving the LMI condition in Eq. (23), the observer gain matrix G is derived as:

$$G = (-10.9786 \quad 9.4665 \quad 59.8127 \quad 11.6423 \quad -14.4855 \quad -3.4610)^T \tag{57}$$

Then, by applying Theorem 2 and solving the constrained convex optimization problem, the control gain matrix K is obtained as:

$$K = \begin{pmatrix} -51.3273 & 1.5326 & 3.7102 & 34.5183 & 1.7920 & 1.3725 \\ -162.0656 & -7.9559 & 0.2540 & 124.4227 & 1.3723 & 5.1444 \end{pmatrix} \tag{58}$$

Consequently, the upper bound on the quadratic cost function $E[\mathcal{J}^*(x_e(0))]$ is obtained as $E[\mathcal{J}^*(x_e(0))] = 6.4975\times10^3$. This simulation compares the results of the proposed method, the conventional linear quadratic regulator (LQR), and the work of Oya et al. [10]. Oya et al. [10] proposed a method for designing observer-based quadratic guaranteed cost control for uncertain systems. The control gain matrix K in Eq. (59) is the result obtained by using the LQR:

$$K = \begin{pmatrix} -0.5272 & 0.2315 & 0.2673 & 0.4336 & 0.1570 & 0.0223 \\ -0.3295 & 0.0495 & 0.0857 & 0.6478 & 0.0223 & 0.1270 \end{pmatrix} \tag{59}$$

H_r in Eq. (60) and K_r in Eq. (61) are the observer gain matrix and the control gain matrix obtained from the design method in the work of Oya et al. [10]:

$$H_r = (-4.7663 \quad 4.7328 \quad 33.2626 \quad 6.0968 \quad -6.2544 \quad -1.2644)^T \tag{60}$$

$$K_r = \begin{pmatrix} -9.6020 & 27.1407 & 15.0468 & 6.3814 & 3.2967 & 0.8400 \\ -4.0787 & 6.3076 & 3.9832 & 5.2499 & 0.8400 & 0.6510 \end{pmatrix} \tag{61}$$

Figs. 2-5 and Figs. 6-9 show the results of the LQR and the method of Oya et al. [10]. From the results, the LQR and the method of Oya et al. [10] did not achieve asymptotical stability. Figs. 6 and 7 show that the state diverged in Case 1 by the method of Oya et al. [10]. Control gain variation was not considered in the work of Oya et al. [10], and thus the system could not be stabilized.

Table 1 State variables of example aircraft

$x_1(t)$	Dimensionless slide-slip velocity (DSV)
$x_2(t)$	Roll
$x_3(t)$	Roll rate
$x_4(t)$	Yaw rate
$x_5(t)$	Aileron angle
$x_6(t)$	Rudder angle

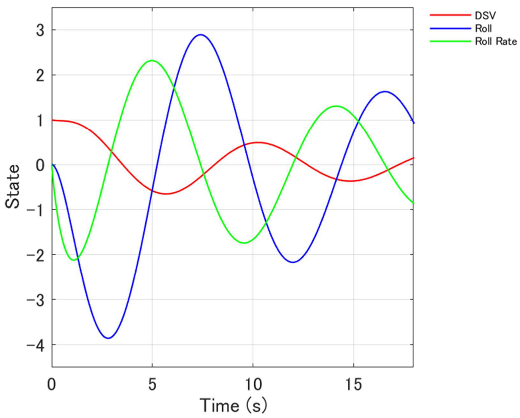

Fig. 2 Time histories of $x_1(t) - x_3(t)$ by LQR (Case 1)

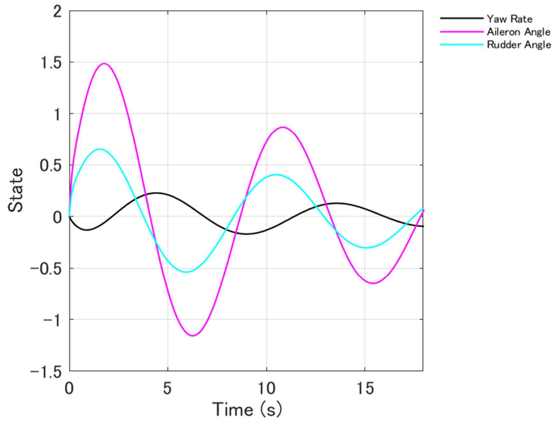

Fig. 3 Time histories of $x_4(t) - x_6(t)$ by LQR (Case 1)

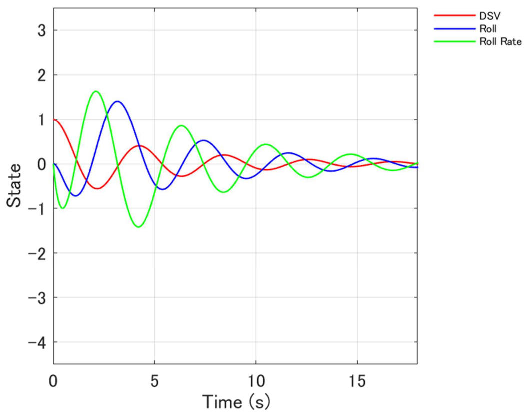

Fig. 4 Time histories of $x_1(t) - x_3(t)$ by LQR (Case 2)

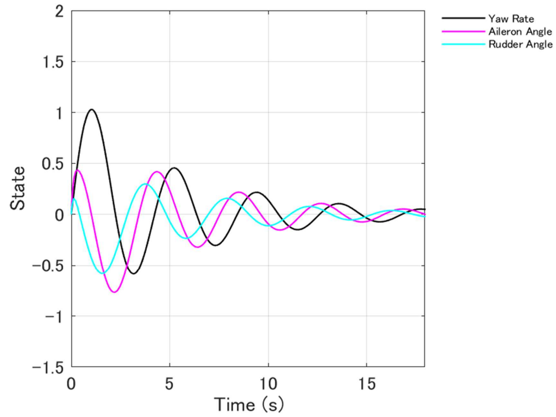

Fig. 5 Time histories of $x_4(t) - x_6(t)$ by LQR (Case 2)

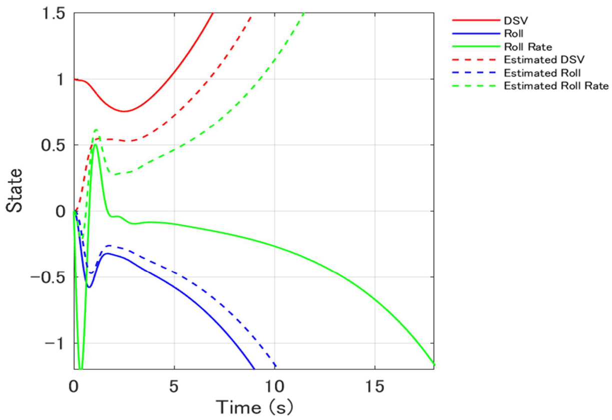

Fig. 6 Time histories of $x_1(t) - x_3(t)$ by the method of Oya et al. [10] (Case 1)

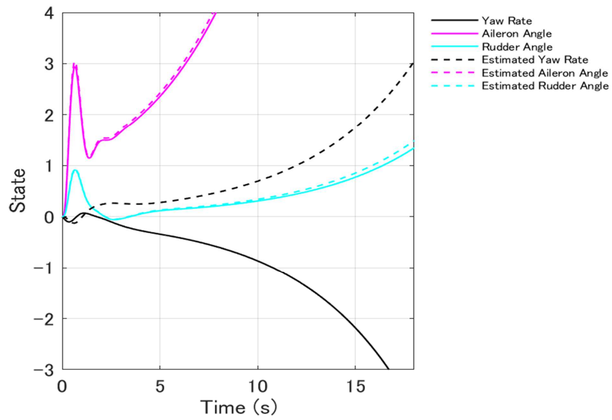

Fig. 7 Time histories of $x_4(t) - x_6(t)$ by the method of Oya et al. [10] (Case 1)

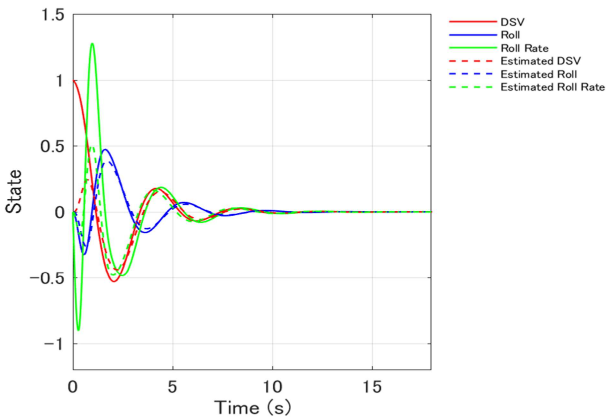

Fig. 8 Time histories of $x_1(t) - x_3(t)$ by the method of Oya et al. [10] (Case 2)

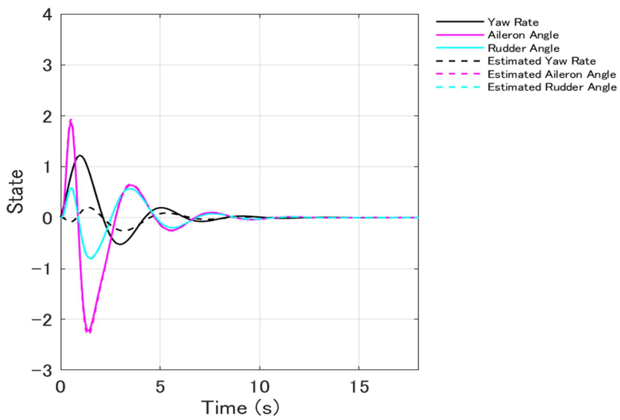

Fig. 9 Time histories of $x_4(t) - x_6(t)$ by the method of Oya et al. [10] (Case 2)

On the other hand, Figs. 10-14 show the results for the proposed controller design method. As shown, even in the presence of system uncertainties and control gain variation, the proposed method achieved asymptotic stability. This demonstrates the effectiveness of the proposed quadratic guaranteed cost controller.

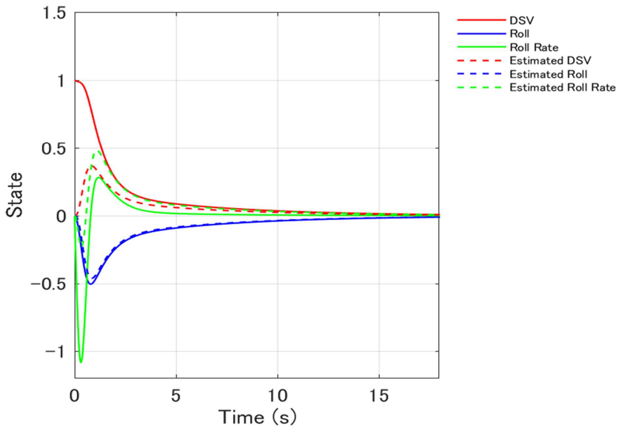

Fig. 10 Time histories of $x_1(t) - x_3(t)$ by the proposed method (Case 1)

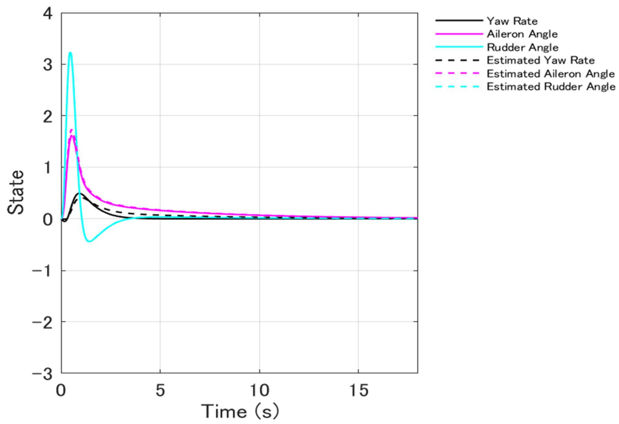

Fig. 11 Time histories of $x_4(t) - x_6(t)$ by the proposed method (Case 1)

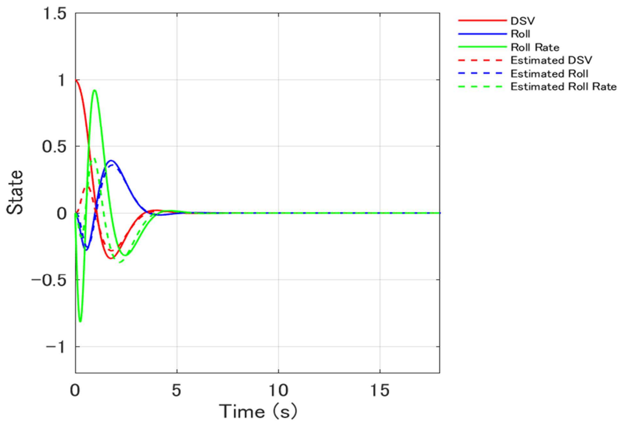

Fig. 12 Time histories of $x_1(t) - x_3(t)$ by the proposed method (Case 2)

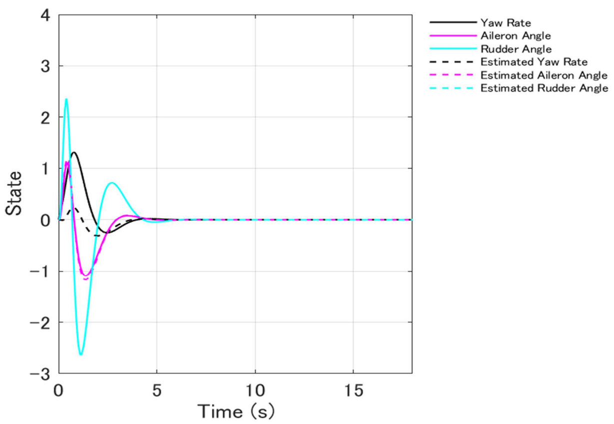

Fig. 13 Time histories of $x_4(t) - x_6(t)$ by the proposed method (Case 2)

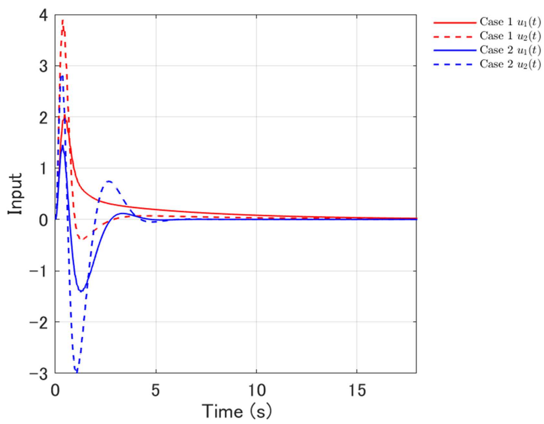

Fig. 14 Time histories of the input by the proposed method

6. Conclusions

This study proposed a method for designing an observer-based quadratic guaranteed cost controller for linear uncertain systems with control gain variation. In the proposed approach, the observer gain matrix was first designed, and then the control gain matrix was determined. The design parameter δ was introduced. The design of an observer-based quadratic guaranteed cost controller was reduced to an LMI condition. Moreover, a robust sub-optimal guaranteed cost controller was investigated. The results of this study are a natural extension of those in the work of Oya et al. [10]. Although the uncertainty in the input matrix has been considered in the work of Oya et al. [10], the proposed design method can be easily applied to such a problem. By introducing additional actuator dynamics and constituting an augmented system, the uncertainties in the input matrix are embedded in the system matrix of the augmented system.

In future work, the proposed adaptive robust controller synthesis will be extended to a broad class of systems, including uncertain linear systems with time delays and decentralized control for large-scale interconnected systems. In the proposed design method, if the parameter δ is set to a larger value, the result will be more conservative. Therefore, reducing conservatism should also be considered.

Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] H. Oya, et al., “Observer-Based Robust Control Giving Consideration to Transient Behavior for Linear Systems with Structured Uncertainties,” International Journal of Control, vol. 75, no. 15, pp. 1231-1240, October 2002.

[2] H. Oya, et al., “Adaptive Robust Control Scheme for Linear Systems with Structure Uncertainties,” IEICE Transactions on Fundamentals of Electronics, Communications, and Computer Sciences, vol. E87-A, no. 8, pp. 2168-2173, August 2004.

[3] H. Oya, et al., “Robust Control Giving Consideration to Time Response for a Linear Systems with Uncertainties,” Transactions of the Institute of Systems, Control, and Information Engineers, vol. 15, no. 8, pp. 404-412, August 2002.

[4] S. S. Chang, et al., “Adaptive Guaranteed Cost Control of Systems with Uncertain Parameters,” IEEE Transactions on Automatic Control, vol. 17, no. 4, pp. 474-483, August 1972.

[5] S. O. R. Moheimani, et al., “Optimal Quadratic Guaranteed Cost Control of a Class of Uncertain Time-Delay Systems,” IEE Proceedings—Control Theory and Applications, vol. 144, no. 2, pp. 183-188, March 1997.

[6] I. R. Petersen, et al., “Optimal Guaranteed Cost Control and Filtering for Uncertain Linear Systems,” IEE Transactions on Automatic Control, vol. 39, no. 9, pp.1971-1977, September 1994.

[7] L. Yu, et al., “An LMI Approach to Guaranteed Cost Control of Linear Uncertain Time Delay Systems,” Automatica, vol. 35, no. 6, pp. 1155-1159, June 1999.

[8] S. H. Park, et al., “H∞ Control with Performance Bound for a Class of Uncertain Linear Systems,” Automatica, vol. 30, no. 12, pp. 2009-2012, April 1994.

[9] R. Petersen, “A Riccati Equation Approach to the Design of Stabilizing Controllers and Observers for a Class of Uncertain Linear Systems,” IEEE Transactions on Automatic Control, vol. 30, no. 9, pp. 904-907, September 1985.

[10] H. Oya, et al., “Observer-Based Guaranteed Cost Control for Polytopic Uncertain Systems,” The Japan Society of Mechanical Engineers, vol. 71, no. 710, pp. 89-98, October 2005.

[11] S. Nagai, et al., “A Point Memory Observer with Adjustable Parameters for a Class of Uncertain Linear Systems with State Delay,” Proceedings of Engineering and Technology Innovation, vol. 11, pp. 38-45, January 2019.

[12] L. H. Keel, et al., “Robust, Fragile, or Optimal?” IEEE Transactions on Automatic Control, vol. 42, no. 8, pp. 1098-1105, August 1997.

[13] G. H. Yang, et al., “H∞ Control for Linear Systems with Additive Controller Gain Variations,” International Journal of Control, vol. 73, no. 16, pp. 1500-1506, February 2000.

[14] D. Famularo, et al., “Robust Non-Fragile LQ Controllers: The Static State Feedback Case,” Proceedings of the 1998 American Control Conference, vol. 2, pp. 1109-1113, June 1998.

[15] H. Oya, et al., “Guaranteed Cost Control for Uncertain Linear Continuous-Time Systems under Control Gain Perturbations,” The Japan Society of Mechanical Engineers, vol. 72, no. 713, pp. 72-101, January 2006.

[16] K. Miyakoshi, et al., “Synthesis of Formation Control Systems for Multi-Agent Systems under Control Gain Perturbations,” Advances in Technology Innovation, vol. 5, no. 2, pp. 112-125, April 2020.

[17] D. Rosinová, et al., “Output Feedback Stabilization of Linear Uncertain Discrete Systems with Guaranteed Cost,” 15th Triennial World Congress, vol. 35, no. 1, pp. 211-215, July 2002.

[18] M. Noton, Modern Control Engineering, New York: Pergamon Press, 1972.

[19] S. Nagai, et al., “Synthesis of Decentralized Variable Gain Robust Controllers with Guaranteed L2 Gain Performance for a Class of Uncertain Large-Scale Interconnected Systems,” Journal of Control Science and Engineering, vol. 2015, Article no. 342867, 2015.

Appendix

In this appendix, the extension to $\mathcal{L}_2$ gain performance is discussed. The following uncertain system is considered:

$$\dot{x}(t) = A(\theta)x(t) + Bu(t) + D_x\omega(t) \tag{A1}$$

$$\dot{z}(t) = Cx(t) + D_z\omega(t) \tag{A2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^l$, and $\omega(t) \in \mathbb{R}^p$ are the vectors of the state, input, output, and disturbance, respectively. The following full-state observer is introduced:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + G(t)(z(t) - C\hat{x}(t)) \tag{A3}$$

Let the input $u(t)$, the estimation error $e(t)$ and the augmented vector $x_e(t)$ be $u(t) \triangleq -K(t)\hat{x}(t)$, $e(t) \triangleq x(t) - \hat{x}(t)$, and $x_e(t) \triangleq (\hat{x}(t)\ e(t))^T$, respectively. The augmented system and the estimation error system can be obtained as follows:

$$\dot{e}(t) = (A(\theta) - G(t)C)e(t) + A_e(\theta)\hat{x}(t) + (D_x - G(t)D_z)\omega(t) \tag{A4}$$

$$\dot{x}_e(t) = \Omega(\theta)x_e(t) + \mathcal{D}\omega(t) \tag{A5}$$

$$\Omega(\theta) = \begin{pmatrix} A - BK(t) & G(t)C \\ A_e(\theta) & A(\theta) - G(t)C \end{pmatrix} \tag{A6}$$

$$\mathcal{D}(t) = \begin{pmatrix} G(t)D_z \\ D_x - G(t)D_z \end{pmatrix} \tag{A7}$$

Please refer to the work of Nagai et al. [19] for a definition and a lemma about $\mathcal{L}_2$ gain performance. The observer gain matrix G is designed as Eq. (24) in section 4.1. To design control gain matrix K, the following Lyapunov function and Hamiltonian are defined:

$$\mathcal{V}_K(x_e) \triangleq x_e^T(t)\Lambda x_e(t) \tag{A8}$$

$$H(x_e, t) \triangleq \dot{\mathcal{V}}_K(x_e) + z^T(t)z(t) - (\gamma^*)^2\omega^T(t)\omega(t) \tag{A9}$$

Here, $(\gamma^*)^2 \triangleq \gamma$ and $x(t) \triangleq (I_n\ I_n)\begin{pmatrix}\hat{x}(t) \\ e(t)\end{pmatrix} = \mathcal{T}x_e(t)$ are introduced. Furthermore, the following equation can be obtained:

$$H(x_e, t) = \begin{pmatrix} x_e^T(t) & \omega^T(t) \end{pmatrix} \begin{pmatrix} \Lambda\Omega(\theta) + \Omega^T(\theta)\Lambda + \mathcal{T}^T C^T C \mathcal{T} & \Lambda\mathcal{D}(t) + \mathcal{T}^T C^T D_z \\ \mathcal{D}^T(t)\Lambda + D_z^T C\mathcal{T} & D_z^T D_z - \gamma I_p \end{pmatrix} \begin{pmatrix} x_e(t) \\ \omega(t) \end{pmatrix} \tag{A10}$$

To satisfy $H(x_e, t) < 0$, the following inequality is considered:

$$\begin{pmatrix} \Lambda\Omega(\theta) + \Omega^T(\theta)\Lambda + \mathcal{T}^T C^T C \mathcal{T} & \Lambda\mathcal{D}(t) + \mathcal{T}^T C^T D_z \\ \mathcal{D}^T(t)\Lambda + D_z^T C\mathcal{T} & D_z^T D_z - \gamma I_p \end{pmatrix} < 0, \quad \forall\theta \in \Delta_{\text{vex}} \tag{A11}$$

$\mathcal{S} \triangleq \text{diag}(S, S_e) \triangleq \Lambda^{-1}(S, S_e > 0 \in \mathbb{R}^{n\times n})$ and $\mathcal{W} \triangleq KS$ are defined, then pre- and post-multiplying both sides of Eq. (A11) by $\text{diag}(\mathcal{S}, I_p)$ and using Lemmas 1 and 2, the following theorem is obtained:

Theorem A: The system in Eqs. (A1) and (A2) is asymptotically stable if there exists a solution $\mathcal{S} > 0, \mathcal{W}, \alpha > 0, \beta > 0, \eta > 0, \lambda > 0$ and $\xi > 0$ satisfying the following LMI:

$$\Psi(\theta)=\begin{pmatrix} \Psi_{11} & \Psi_{12}(\theta) & GD_z+SC^TD_z & SC^T & \varepsilon_K S & 0 & 0 & 0 & 0 \\ \Psi_{12}^T(\theta) & \Psi_{22}(\theta) & D_x-GD_z+S_eC^TD_z & S_eC^T & 0 & S_eC^T & S_eC^T & 0 & 0 \\ D_z^TG^T+D_z^TCS & D_x^T-D_z^TG^T+D_z^TCS_e & D_z^TD_z-\gamma I_p & 0 & 0 & 0 & 0 & D_z & D_z \\ CS & CS_e & 0 & -I_l & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_K S & 0 & 0 & 0 & -\alpha I_n & 0 & 0 & 0 & 0 \\ 0 & CS_e & 0 & 0 & 0 & -\beta I_l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\eta I_l & 0 & 0 \\ 0 & 0 & D_z & 0 & 0 & 0 & 0 & -\xi I_l & 0 \\ 0 & 0 & D_z & 0 & 0 & 0 & 0 & 0 & -\lambda I_l \end{pmatrix}<0, \tag{A12}$$

$$\forall\theta\in\Delta_{\mathrm{vex}}$$

$$\Psi_{11}=AS+SA^T-B\mathcal{W}-\mathcal{W}^TB^T+\alpha BB^T+\eta\varepsilon_G^2I_n+\xi\varepsilon_G^2I_n \tag{A13}$$

$$\Psi_{12}(\theta)=SA_e^T(\theta)+GCS_e \tag{A14}$$

$$\Psi_{22}(\theta)=A(\theta)S_e+S_eA^T(\theta)-GCS_e-S_eC^TG^T+\beta\varepsilon_G^2I_n+\lambda\varepsilon_G^2I_n \tag{A15}$$

By solving the LMI in Eq. (23) in advance, the observer gain matrix G is derived as $G = Y_e^{-1}H_e$. If the solution of the LMI condition in Eq. (A12) is obtained, then the control gain matrix K can be computed as $K = \mathcal{W}S^{-1}$. Therefore, the control law $u(t) = -K\hat{x}(t)$ is the observer-based control.