Vibration Analysis of Thermo Elastic Micro Beam with Double Porosity Structure

Rajneesh Kumar[1], Richa Vohra[2, *], and M. G. Gorla[3]

[1]Department of Mathematics, Kurukshetra University, Kurukshetra, Haryana, India.

[2, 3]Department of Mathematics& Statistics, H. P. University, Shimla, HP, India.

Received 03 September 2016; received in revised form 17 December 2016; accepted 06 January 2017

Abstract

The present investigation is concerned with vibration analysis of a homogeneous, isotropic thermo elastic micro beam with double porosity structure subjected to sinusoidal pulse heating. Lord-Shulman [1] theory of thermo elasticity with one relaxation time is used to solve the problem. Laplace transform technique has been used to obtain the expressions for lateral deflection, axial stress, axial displacement, volume fraction field and temperature distribution. A numerical inversion technique has been applied to recover the resulting quantities in the physical domain. Variations of axial displacement, axial stress, lateral deflection, volume fraction field and temperature distribution against axial distance are depicted graphically to show the effect of porosity and relaxation time parameters. Some particular cases are also deduced.

Keywords: Double porosity, thermo elasticity, Lord-Shulman theory, micro beam, sinusoidal pulse heating

1. Introduction

Pores and fractures can be seen in engineering structures due to reasons like erosion, corrosion, fatigue or accidents which affect the dynamic behavior of the entire structure to a considerable extent. This leads to the development of double porosity model which has its applications in geophysics, rock mechanics and many branches of engineering like civil engineering, chemical engineering and the petroleum industry. Biot [2] proposed model for porous media with single porosity. Later on Barenblatt, Zheltovand and Kochina [3] introduced a model for porous media with double porosity structure. The double porosity model consists of two coexisting degrees of porosity in which one corresponds to porous matrix and other corresponds to fissure matrix.

Aifantis [4-6] introduced a multi-porous system and studied the mechanics of diffusion in solids. Wilson and Aifanits [7] presented the theory of consolidation with the double porosity. Khaled, Beskos and Aifantis [8] employed a finite element method to consider the numerical solutions of the differential equation of the theory of consolidation with double porosity developed by Wilson and Aifantis [7]. Beskos and Aifantis [9] presented the theory of consolidation with double porosity-II and obtained the analytical solutions to two boundary value problems. Khaliliand and Selvadurai [10] presented a fully coupled constitutive model for thermo-hydro –mechanical analysis in elastic media with double porosity structure. Various authors [11-13] investigated problems for elastic solids and thermo elastic solids in the theory of thermo elasticity with double porosity based on Darcy's law.

Nunziato and Cowin [14] developed a non-linear theory of elastic material with voids. Later, Cowin and Nunziato [15] developed a theory of linear elastic materials with voids for the mathematical study of the mechanical behavior of porous

*Corresponding author. E-mail address: richavhr88@gmail.com

solids. Iesan and Quintanilla [16] derived a theory of thermo elastic solids with double porosity structure by using the theory developed by Nunziato and Cowin. Darcy's law is not used in developing this theory. So far not much work has been done on the theory of thermo elasticity with double porosity based on the model proposed by Iesan and Quintanilla [16]. Recently, investigations have been started in the theory of thermo elasticity with double porosity [16] which has a significant application in continuum mechanics.

The demand for engineering structures is continuously increasing. Aerospace vehicles, bridges, and automobiles are examples of these structures. Many aspects have to be taken into consideration in the design of these structures to improve their performance and extend their life. One aspect of the design process is the dynamic response of structures. The dynamics of distributed parameter and continuous systems, like beams, were governed by linear and non-linear partial differential equations in space and time.

Micro-scale mechanical resonators have high sensitivity as well as fast response and are widely used as sensors and modulators. Recently, micro- and nano-mechanical resonators have attracted considerable attention due to their many important technological applications. Accurate analysis of various effects on the characteristics of resonators, such as resonant frequencies and quality factors, is crucial for designing high-performance components. The vibration problems of uniform Euler- Bernoulli beams can be solved by analytical or approximate approaches [17, 18]. Boley [19] analyzed the vibrations of a simply supported rectangular beam subjected to a suddenly applied heat input distributed along its span. Manolis and Beskos [20] examined the thermally induced vibration of structures consisting of beams exposed to rapid surface heating. Huniti, Al-Nimrand Naij [21] investigated the thermally induced displacements and stresses of a rod using the Laplace transformation technique. Biondi and Caddemi [22] studied the problem of the integration of the static governing equations of the uniform Euler-Bernoulli beams with discontinuities, considering the flexural stiffness and slope discontinuities. Fang, Sun and Soh [23] analyzed the vibrations in micro beam resonators induced by laser. Sharma and Grover [24] analysed the thermo elastic vibrations in micro-/nano-scale beam resonators with the presence of voids. Esen [25] presented the analysis of transverse and longitudinal vibrations of a thin plate which carries a load moving along an arbitrary trajectory with variable velocity. Dehrouyeh-Semnani, Dehrouyeh, Torabi-Kafshgari and Nikkah-Bahrami [26] discussed the model of damped sandwich beam based on symmetric-deviatoric couple stress theory and investigated the vibration damping characteristics of the micro beam. Dehrouyeh-Semnani, Dehrouyeh, Torabi-Kafshgari and Nikkah-Bahrami [27] analysed the free flexural vibration phenomenon of functionally graded micro beams with geometric imperfection.Mojahedi and Rahaeifard [28] studied a non-linear model for coupled three dimensional micro beam as well as static bending and free vibration analysis of a micro bridge.

In the present paper, vibration analysis of an Euler-Bernoulli thermo elastic micro beam with double porosity structure due to sinusoidal pulse heating is studied. Lord-Shulman theory of thermo elasticity is used to investigate the problem. Laplace transform has been applied to find the expressions for lateral deflection, axial stress, axial displacement, volume fraction fields and temperature distribution. The resulting quantities are obtained in the physical domain by using a numerical inversion technique. Variations of axial displacement, axial stress, lateral deflection, and volume fraction field and temperature distribution against axial distance are depicted graphically to show the effect of porosity and thermal relaxation time. Some special cases of interest have also been deduced. The problem has a great significance in many branches of engineering like soil engineering, civil engineering etc. The double porosity model appears in several areas of mechanics, such as composition and behavior of bones as well as some phenomenon of geophysics. The intended applications of this theory are to geological materials such as rocks and soils and to manufactured porous materials such as ceramics and pressed powders. Study of vibration analysis of micro beam for such model has many applications in real world where the interest is in various phenomena occurring in earthquakes and measurement of displacements, stresses and temperature field due to the presence of certain sources.

2. Basic Equations

Following Iesan and Quintanilla [16] and Lord and Shulman [1]; the constitutive relations and field equations for homogeneous isotropic thermo elastic material with double porosity structure in the absence of body forces, extrinsic equilibrated body forces and heat sources can be written as:

2.1. Equations of motion:

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{j,ji} + b\varphi_{,i} + d\psi_{,i} - \beta T_{,i} = \rho\ddot{u}_i, \tag{1}$$

2.2. Equilibrated Stress Equations of motion:

$$\alpha\nabla^2\varphi + b_1\nabla^2\psi - bu_{r,r} - \alpha_1\varphi - \alpha_3\psi + \gamma_1 T = \kappa_1\ddot{\varphi}, \tag{2}$$

$$b_1\nabla^2\varphi + \gamma\nabla^2\psi - du_{r,r} - \alpha_3\varphi - \alpha_2\psi + \gamma_2 T = \kappa_2\ddot{\psi}, \tag{3}$$

2.3. Equation of heat conduction:

$$\left(1+\tau_0\frac{\partial}{\partial t}\right)\left(\beta T_0\dot{u}_{j,j} + \gamma_1 T_0\dot{\varphi} + \gamma_2 T_0\dot{\psi} + \rho C^*\dot{T}\right) = K^*\nabla^2 T \tag{4}$$

where λ and μ are Lame's constants, ρ is the mass density; $\beta = (3\lambda+2\mu)\alpha_t$; α_t is the linear thermal expansion; C^* is the specific heat at constant strain, u_i is the displacement components; t_{ij} is the stress tensor; κ_1 and κ_2 are coefficients of equilibrated inertia; φ is the volume fraction field corresponding to pores and ψ is the volume fraction field corresponding to fissures; K^* is the coefficient of thermal conductivity; τ_0 is the thermal relaxation time, κ_1 and κ_2 are coefficients of equilibrated inertia and $b, d, b_1\ \gamma, \gamma_1, \gamma_2$ are constitutive coefficients; δ_{ij} is the Kronecker's delta; T is the temperature change measured form the absolute temperature $T_0\ (T_0 \neq 0)$; a superposed dot represents differentiation with respect to time variable t .

3. Formulation of the problem

Let us consider a thermo elastic micro beam with double porosity structure along the axial direction (x -axis) of the

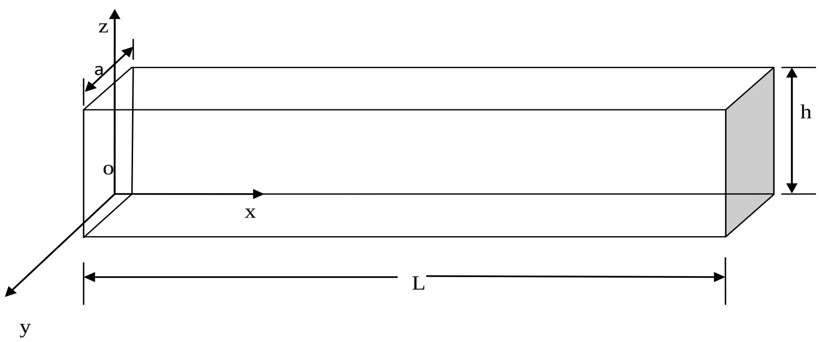

Fig. 1 Geometry of the beam

beam. The beam has cross-sectional area A , moment of inertia I , length L , width a and thickness h as shown in the Fig. 1.

The micro beam undergoes bending vibrations of small amplitude about the x -axis such that the deflection is consistent with the linear Euler-Bernoulli theory. Therefore, the displacements can be written as

$$u_1 = u = -z\frac{\partial w}{\partial x}, \quad u_2 = 0, \quad u_3 = w(x,t) \tag{5}$$

where w is the lateral deflection and u is the axial displacement. The one-dimensional constitutive equation can be written as

$$t_x = -z(\lambda+2\mu)\frac{\partial^2 w}{\partial x^2} + b\varphi + d\psi - \beta T \tag{6}$$

where t_x is the axial stress. The equation of motion of free flexural vibrations of the beam is given by

$$\frac{\partial^2 M}{\partial x^2} + \rho A\left(\frac{\partial^2 w}{\partial t^2}\right) = 0 \tag{7}$$

where $A = ah$ is the cross-section area and M is the flexural moment of cross section of micro beam. The flexural moment of the cross section of the beam, with the aid of Eq. (6) is given by

$$M(x,t) = -a\int_{-\frac{h}{2}}^{\frac{h}{2}} t_x z dz = (\lambda+2\mu) I\frac{\partial^2 w}{\partial x^2} - M_\varphi - M_\psi + M_T \tag{8}$$

where $I = ah^3/12$ is the moment of inertia of the cross-section and M_φ, M_ψ are the volume fraction field moments and M_T is thermal moment of the beam and are given by

$$M_\varphi = b\int_{-\frac{h}{2}}^{\frac{h}{2}} a\varphi z dz, M_\psi = d\int_{-\frac{h}{2}}^{\frac{h}{2}} a\psi z dz, M_T = \beta\int_{-\frac{h}{2}}^{\frac{h}{2}} aTz dz \tag{9}$$

Substituting Eq. (8) in Eq. (7), we get the equation of motion of the micro beam as

$$(\lambda+2\mu) I\frac{\partial^4 w}{\partial x^4} + \rho A\left(\frac{\partial^2 w}{\partial t^2}\right) - \frac{\partial^2 M_\varphi}{\partial x^2} - \frac{\partial^2 M_\psi}{\partial x^2} + \frac{\partial^2 M_T}{\partial x^2} = 0 \tag{10}$$

Eqs. (2)-(4) with the aid of Eq. (5) can be written as

$$\alpha\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2}\right) + b_1\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right) + bz\frac{\partial^2 w}{\partial x^2} - \alpha_1\varphi - \alpha_3\psi + \gamma_1 T = \kappa_1\frac{\partial^2\varphi}{\partial t^2} \tag{11}$$

$$b_1\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2}\right) + \gamma\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right) + dz\frac{\partial^2 w}{\partial x^2} - \alpha_3\varphi - \alpha_2\psi + \gamma_2 T = \kappa_2\frac{\partial^2\psi}{\partial t^2} \tag{12}$$

$$K^*\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) = \left(1+\tau_0\frac{\partial}{\partial t}\right)\left[-\beta T_0 z\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right) + \gamma_1 T_0\dot{\varphi} + \gamma_2 T_0\dot{\psi} + \rho C^*\dot{T}\right] \tag{13}$$

4. Solution of the problem

For the present micro beam, we assume that there is no flow of heat and volume fraction fields across the surfaces $(z = \pm h/2)$ so that $\partial T/\partial z = \partial\varphi/\partial z = \partial\psi/\partial z = 0$ at $z = \pm h/2$. For a very thin beam, assuming that volume fraction fields and temperature increment in terms of $\sin(\pi z/h)$ function along the thickness direction. Therefore,

$$\begin{aligned} \varphi(x,z,t) &= \Phi(x,t)\sin(\pi z/h) \\ \psi(x,z,t) &= \Psi(x,t)\sin(\pi z/h) \\ T(x,z,t) &= \Theta(x,t)\sin(\pi z/h) \end{aligned} \tag{14}$$

Substituting Eq. (14) in Eqs. (11)-(13), and then into Eq. (10) yields

$$(\lambda+2\mu) I\frac{\partial^4 w}{\partial x^4} + \rho ah\left(\frac{\partial^2 w}{\partial t^2}\right) - \frac{2abh^2}{\pi^2}\frac{\partial^2\Phi}{\partial x^2} - \frac{2adh^2}{\pi^2}\frac{\partial^2\Psi}{\partial x^2} + \frac{2a\beta h^2}{\pi^2}\frac{\partial^2\Theta}{\partial x^2} = 0 \tag{15}$$

Multiplying Eqs. (11)-(13) by z and integrating them with respect to z within the limits $-h/2$ to $h/2$, we get

$$\alpha\left(\frac{\partial^2\Phi}{\partial x^2}-\frac{\pi^2\Phi}{h^2}\right)+b_1\left(\frac{\partial^2\Psi}{\partial x^2}-\frac{\pi^2\Psi}{h^2}\right)+\frac{b\pi^2 h}{24}\frac{\partial^2 w}{\partial x^2}-\alpha_1\Phi-\alpha_3\Psi+\gamma_1\Theta=\kappa_1\frac{\partial^2\Phi}{\partial t^2} \tag{16}$$

$$b_1\left(\frac{\partial^2\Phi}{\partial x^2}-\frac{\pi^2\Phi}{h^2}\right)+\gamma\left(\frac{\partial^2\Psi}{\partial x^2}-\frac{\pi^2\Psi}{h^2}\right)+\frac{d\pi^2 h}{24}\frac{\partial^2 w}{\partial x^2}-\alpha_3\Phi-\alpha_2\Psi+\gamma_2\Theta=\kappa_2\frac{\partial^2\Psi}{\partial t^2} \tag{17}$$

$$K^*\left(\frac{\partial^2\Theta}{\partial x^2}-\frac{\pi^2\Theta}{h^2}\right)=\left(1+\tau_0\frac{\partial}{\partial t}\right)\left[-\frac{\beta T_0\pi^2 h}{24}\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right)+\gamma_1 T_0\frac{\partial\Phi}{\partial t}+\gamma_2 T_0\frac{\partial\Psi}{\partial t}+\rho C^*\frac{\partial\Theta}{\partial t}\right] \tag{18}$$

Introducing non-dimensional variables as

$$\begin{aligned}&x' = \frac{1}{L}x,\ u' = \frac{1}{L}u,\ w' = \frac{1}{L}w,\ t'_x = \frac{t_x}{E},\ \Phi' = \frac{L}{\alpha}\Phi,\\ &\Theta' = \frac{\beta}{E}\Theta,\ t' = \frac{c_1}{L}t,\ \tau'_0 = \frac{c_1}{L}\tau_0,\ \Psi' = \frac{L}{\alpha}\Psi\end{aligned} \tag{19}$$

where $c_1^2 = \frac{\lambda+2\mu}{\rho}$ and $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is Young's Modulus. Making use of Eqs. (15)-(18), we obtain

$$\frac{\partial^4 w}{\partial x^4}+a_1\left(\frac{\partial^2 w}{\partial t^2}\right)-a_2\frac{\partial^2\Phi}{\partial x^2}-a_3\frac{\partial^2\Psi}{\partial x^2}+a_4\frac{\partial^2\Theta}{\partial x^2}=0 \tag{20}$$

$$a_5\frac{\partial^2\Phi}{\partial x^2}-a_6\Phi+a_7\frac{\partial^2\Psi}{\partial x^2}-a_8\Psi+a_9\frac{\partial^2 w}{\partial x^2}-a_{10}\Phi-a_{11}\Psi+a_{12}\Theta-\frac{\partial^2\Phi}{\partial t^2}=0 \tag{21}$$

$$a_{13}\frac{\partial^2\Phi}{\partial x^2}-a_{14}\Phi+a_{15}\frac{\partial^2\Psi}{\partial x^2}-a_{16}\Psi+a_{17}\frac{\partial^2 w}{\partial x^2}-a_{18}\Phi-a_{19}\Psi+a_{20}\Theta-\frac{\partial^2\Psi}{\partial t^2}=0 \tag{22}$$

$$\frac{\partial^2\Theta}{\partial x^2}-a_{21}\Theta=\left(1+\tau_0\frac{\partial}{\partial t}\right)\left[a_{22}\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right)+a_{23}\frac{\partial\Phi}{\partial t}+a_{24}\frac{\partial\Psi}{\partial t}+a_{25}\frac{\partial\Theta}{\partial t}\right] \tag{23}$$

where

$$\begin{aligned}&a_1=\frac{\rho a h c_1^2 L^2}{I(\lambda+2\mu)},\ a_2=\frac{2ab\alpha h^2}{I\pi^2(\lambda+2\mu)},\ a_3=\frac{2ad\alpha h^2}{I\pi^2(\lambda+2\mu)},\ a_4=\frac{2ah^2EL}{I\pi^2(\lambda+2\mu)},\ a_5=\frac{\alpha}{k_1c_1^2},\\ &a_6=\frac{\alpha\pi^2L^2}{k_1c_1^2h^2},\ a_7=\frac{b_1}{k_1c_1^2},\ a_8=\frac{b_1\pi^2L^2}{k_1c_1^2h^2},\ a_9=\frac{bh\pi^2L^2}{24\alpha k_1c_1^2},\ a_{10}=\frac{\alpha_1L^2}{k_1c_1^2},\ a_{11}=\frac{\alpha_3L^2}{k_1c_1^2},\\ &a_{12}=\frac{\gamma_1EL^3}{\alpha\beta k_1c_1^2},\ a_{13}=\frac{b_1}{k_2c_1^2},\ a_{14}=\frac{b_1\pi^2L^2}{k_2c_1^2h^2},\ a_{15}=\frac{\gamma}{k_2c_1^2},\ a_{16}=\frac{\gamma\pi^2L^2}{k_2c_1^2h^2},\ a_{17}=\frac{dh\pi^2L^2}{24\alpha k_2c_1^2},\\ &a_{18}=\frac{\alpha_3L^2}{k_2c_1^2},\ a_{19}=\frac{\alpha_2L^2}{k_2c_1^2},\ a_{20}=\frac{\gamma_2EL^3}{\alpha\beta k_2c_1^2},\ a_{21}=\frac{\pi^2L^2}{h^2},\ a_{22}=-\frac{T_0hc_1\pi^2\beta^2}{24EK^*},\\ &a_{23}=\frac{\alpha\beta T_0\gamma_1c_1}{EK^*},\ a_{24}=\frac{\alpha\beta T_0\gamma_2c_1}{EK^*},\ a_{25}=\frac{\rho\, C^*c_1L}{K^*}\end{aligned} \tag{24}$$

5. Initial and boundary conditions

The initial conditions of the problem are assumed to be homogeneous and are taken as

$$w(x,t)\big|_{t=0} = \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = 0,\ \ \Phi(x,t)\big|_{t=0} = \left.\frac{\partial \Phi(x,t)}{\partial t}\right|_{t=0} = 0,$$
$$\Psi(x,t)\big|_{t=0} = \left.\frac{\partial \Psi(x,t)}{\partial t}\right|_{t=0} = 0,\ \ \Theta(x,t)\big|_{t=0} = \left.\frac{\partial \Theta(x,t)}{\partial t}\right|_{t=0} = 0 \tag{25}$$

These initial conditions are supplemented by considering that the two ends of the micro beam are clamped:

$$w(x,t)\big|_{x=0.L} = \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0,L} = 0 \tag{26}$$

The micro beam is thermally loaded by sinusoidal pulse heating incidents into the surface of the micro beam $x=0$ with pulse width t_0 as

$$\Theta(x,t)\big|_{x=0} = \begin{cases} \sin\dfrac{\pi t}{t_0} & 0 \le t \le t_0 \\ 0 & t > t_0,\ t < 0 \end{cases} \tag{27}$$

We also assume that the volume fraction fields and the temperature should satisfy the following relation:

$$\Phi(x,t)\big|_{x=0} = 0,\ \ \left.\frac{\partial \Phi}{\partial x}\right|_{x=L} = 0,\ \ \Psi(x,t)\big|_{x=0} = 0,\ \ \left.\frac{\partial \Psi}{\partial x}\right|_{x=L} = 0,\ \ \left.\frac{\partial \Theta}{\partial x}\right|_{x=L} = 0 \tag{28}$$

6. Solution in the Laplace transform domain

Applying the Laplace transform defined by

$$\overline{f}(s) = L[f(t)] = \int_0^{\infty} f(t)e^{-st}dt \tag{29}$$

to the Eqs. (20)-(23) under the initial conitions (25), after some simplifications, we obtain

$$\left[\frac{d^{10}}{dx^{10}} + B_1\frac{d^8}{dx^8} + B_2\frac{d^6}{dx^6} + B_3\frac{d^4}{dx^4} + B_4\frac{d^2}{dx^2} + B_5\right]\left(\bar{w},\bar{\Phi},\bar{\Psi},\bar{\Theta}\right) = 0 \tag{30}$$

B_1, B_2, B_3, B_4, B_5, are given in the appendix I. The solution of the Eq. (31), in the Laplace transform domain can be written as

$$\left(\bar{w},\bar{\Phi},\bar{\Psi},\bar{\Theta}\right) = \sum_{i=1}^{5}\left(1, g_{1i}, g_{2i}, g_{3i}\right)\left(D_i e^{-\lambda_i x} + D_{i+5}e^{\lambda_i x}\right) \tag{31}$$

g_{1i}, g_{2i}, g_{3i}; $i = 1,2,3,4,5$ are given in the appendix II. Here $\pm\lambda_i$, $i = 1,2,3,4,5$ are the roots of the characteristic equation

$$\lambda^{10} + B_1\lambda^8 + B_2\lambda^6 + B_3\lambda^4 + B_4\lambda^2 + B_5 = 0 \tag{32}$$

Making use of Eq. (31) in Eq. (5) and with the aid of Eqs. (14) and (19), we obtain the corresponding expressions for axial displacement and axial stress in the Laplace transform domain as

$$\bar{u} = -z\frac{d\bar{w}}{dx} = -z\sum_{i=1}^{5}\left(-\lambda_i D_i e^{-\lambda_i x} + \lambda_i D_{i+5}e^{\lambda_i x}\right) \tag{33}$$

$$\bar{t}_x = \sum_{i=1}^{5}\left[P_1 z\lambda_i^2 + \sin\left(\pi z/h\right)(P_2 g_{1i} + P_3 g_{2i} - g_{3i})\right]\left(D_i e^{-\lambda_i x} + D_{i+5}e^{\lambda_i x}\right) \tag{34}$$

where $P_1 = -\dfrac{(\lambda+2\mu)}{E}$, $P_2 = \dfrac{b\alpha}{EL}$, $P_3 = \dfrac{d\alpha}{EL}$, the boundary conditions (26)-(28) in the Laplace transform domain take the form as

$$\begin{aligned} &\bar{w}(x,s)\big|_{x=0.L} = \frac{d\bar{w}(x,s)}{\partial x}\bigg|_{x=0,L} = 0, \quad \bar{\Theta}(x,s)\big|_{x=0} = \frac{\pi t_0}{\pi^2 + t_0^2 s^2} = \bar{F}(s), \\ &\bar{\Phi}(x,s)\big|_{x=0} = 0, \quad \frac{\partial\bar{\Phi}}{\partial x}\bigg|_{x=L} = 0, \quad \bar{\Psi}(x,s)\big|_{x=0} = 0, \quad \frac{\partial\bar{\Psi}}{\partial x}\bigg|_{x=L} = 0, \quad \frac{\partial\bar{\Theta}}{\partial x}\bigg|_{x=L} = 0 \end{aligned} \tag{35}$$

In order to determine the unknown parameters, substituting Eq. (31) in the boundary conditions (35), we obtain a system of ten linear equations in the matrix form as

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
e^{-\lambda_1 L} & e^{-\lambda_2 L} & e^{-\lambda_3 L} & e^{-\lambda_4 L} & e^{-\lambda_5 L} & e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} & e^{\lambda_4 L} & e^{\lambda_5 L} \\
-\lambda_1 & -\lambda_2 & -\lambda_3 & -\lambda_4 & -\lambda_5 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
-\lambda_1 e^{-\lambda_1 L} & -\lambda_2 e^{-\lambda_2 L} & -\lambda_3 e^{-\lambda_3 L} & -\lambda_4 e^{-\lambda_4 L} & -\lambda_5 e^{-\lambda_5 L} & \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} & \lambda_4 e^{\lambda_4 L} & \lambda_5 e^{\lambda_5 L} \\
g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{11} & g_{12} & g_{13} & g_{14} & g_{15} \\
-g_{11}\lambda_1 e^{-\lambda_1 L} & -g_{12}\lambda_2 e^{-\lambda_2 L} & -g_{13}\lambda_3 e^{-\lambda_3 L} & -g_{14}\lambda_4 e^{-\lambda_4 L} & -g_{15}\lambda_5 e^{-\lambda_5 L} & g_{11}\lambda_1 e^{\lambda_1 L} & g_{12}\lambda_2 e^{\lambda_2 L} & g_{13}\lambda_3 e^{\lambda_3 L} & g_{14}\lambda_4 e^{\lambda_4 L} & g_{15}\lambda_5 e^{\lambda_5 L} \\
g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{21} & g_{22} & g_{23} & g_{24} & g_{25} \\
-g_{21}\lambda_1 e^{-\lambda_1 L} & -g_{22}\lambda_2 e^{-\lambda_2 L} & -g_{23}\lambda_3 e^{-\lambda_3 L} & -g_{24}\lambda_4 e^{-\lambda_4 L} & -g_{25}\lambda_5 e^{-\lambda_5 L} & g_{21}\lambda_1 e^{\lambda_1 L} & g_{22}\lambda_2 e^{\lambda_2 L} & g_{23}\lambda_3 e^{\lambda_3 L} & g_{24}\lambda_4 e^{\lambda_4 L} & g_{25}\lambda_5 e^{\lambda_5 L} \\
g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{31} & g_{32} & g_{33} & g_{34} & g_{35} \\
-g_{31}\lambda_1 e^{-\lambda_1 L} & -g_{32}\lambda_2 e^{-\lambda_2 L} & -g_{33}\lambda_3 e^{-\lambda_3 L} & -g_{34}\lambda_4 e^{-\lambda_4 L} & -g_{35}\lambda_5 e^{-\lambda_5 L} & g_{31}\lambda_1 e^{\lambda_1 L} & g_{32}\lambda_2 e^{\lambda_2 L} & g_{33}\lambda_3 e^{\lambda_3 L} & g_{34}\lambda_4 e^{\lambda_4 L} & g_{35}\lambda_5 e^{\lambda_5 L}
\end{bmatrix}
\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \\ D_{10} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \bar{F}(s) \\ 0 \end{bmatrix} \tag{36}$$

On solving the above system of Eq. (36), we obtain the values of unknown parameters $D_i, i = 1, 2, \ldots, 10$. This completes the solution of the problem in Laplace transform domain.

7. Particular cases

With the help of the problem investigated in the present paper, we can also deduce some problems by setting the values of some parameters equal to zero in the present problem which is discussed in detail. Some of the particular cases which we have deduced from the present investigation are as follows:

Case 7.1 If $b_1 = \alpha_3 = \gamma = \alpha_2 = \gamma_2 = d \to 0$ in Eqs. (1)-(4), along with Eq. (6), then the corresponding basic equations and the one dimensional constitutive relation takes the form as

$$\mu\nabla^2 u_i + (\lambda+\mu) u_{j,ji} + b\varphi_{,i} - \beta T_{,i} = \rho\ddot{u}_i \tag{37}$$

$$\alpha\nabla^2\varphi - bu_{r,r} - \alpha_1\varphi + \gamma_1 T = \kappa_1\ddot{\varphi} \tag{38}$$

$$\left(1+\tau_0\frac{\partial}{\partial t}\right)\left(\beta T_0\dot{u}_{j,j} + \gamma_1 T_0\dot{\varphi} + \rho C^*\dot{T}\right) = K^*\nabla^2 T \tag{39}$$

$$t_x = -z(\lambda+2\mu)\frac{\partial^2 w}{\partial x^2} + b\varphi - \beta T \tag{40}$$

On solving the above system of Eqs. (37)-(40), we obtain the corresponding expressions for a thermo elastic micro beam with single porosity.

Case 7.2 If $\tau_0 = 0$ in Eq. (4), then the corresponding heat conduction equation in context of coupled theory (CT) of thermo elasticity becomes

$$\beta T_0\dot{u}_{j,j} + \gamma_1 T_0\dot{\varphi} + \gamma_2 T_0\dot{\psi} + \rho C^*\dot{T} = K^*\nabla^2 T \tag{41}$$

On solving Eqs. (1)-(3), (6) and (41), we obtain the corresponding expressions for a thermo elastic micro beam with double porosity in context of coupled theory (CT) of thermo elasticity.

8. Inversion of the Laplace domain

To determine the displacement, stresses and temperature distribution in the physical domain, we will adopt a numerical inversion method given by [29].

In this method, Laplace domain $\bar{f}(s)$ can be inverted to time domain $f(t)$ as

$$f(t)=\frac{e^{\Omega t}}{t_1}\left[\frac{1}{2}\bar{f}(\Omega)+\operatorname{Re}\sum_{k=1}^{N}\bar{f}\left(\Omega+\frac{\iota k\pi}{t_1}\right)\exp\left(\frac{\iota k\pi t}{t_1}\right)\right],\quad 0<t_1<2t \tag{42}$$

where Re is the real part and ι is the imaginary number unit. The value of N is chosen sufficiently large and it represents the number of terms in the truncated Fourier series such that $f(t)=\exp(\Omega t)\operatorname{Re}\left[\bar{f}\left(\Omega+\frac{\iota N\pi}{t_1}\right)\exp\left(\frac{\iota N\pi t}{t_1}\right)\right]\le\varepsilon_1$, ε_1 is a prescribed small positive number. Also, the value of Ω should satisfy the relation $\Omega t\approx 4.7$ for the faster convergence [30].

9. Numerical results and discussion

For the purpose of numerical computation, a beam made of copper like material is analyzed. The material parameters are as shown in Table 1.

Table 1 The aspect ratio of the beam is fixed as $L/h=10$, $a/h=0.5$, $x_3=h/6$

Parameter	Value	Parameter	Value
λ	$7.76\times10^{10}\,Nm^{-2}$	K^*	$3.86\times10^{3}\,Ns^{-1}K^{-1}$
C^*	$3.831\times10^{3}\,m^2s^{-2}K^{-1}$	T_0	$293\,K$
μ	$3.86\times10^{10}\,Nm^{-2}$	ρ	$8.954\times10^{3}\,Kgm^{-3}$
α_t	$1.78\times10^{-5}\,K^{-1}$	α_2	$2.4\times10^{10}\,Nm^{-2}$
t_0	$0.1s$	α_3	$2.5\times10^{10}\,Nm^{-2}$
v	0.22	γ	$1.1\times10^{-5}\,N$
t	$0.12s$	α	$1.3\times10^{-5}\,N$
γ_1	$0.16\times10^{5}\,Nm^{-2}$	b_1	$0.12\times10^{-5}\,N$
d	$0.1\times10^{10}\,Nm^{-2}$	γ_2	$0.219\times10^{5}\,Nm^{-2}$
κ_1	$0.1456\times10^{-12}\,Nm^{-2}s^{2}$	κ_2	$0.1546\times10^{-12}\,Nm^{-2}s^{2}$
b	$0.9\times10^{10}\,Nm^{-2}$	α_1	$2.3\times10^{10}\,Nm^{-2}$

The software MATLAB has been used to find the values of lateral deflection, axial stress, axial displacement, volume fraction field and temperature distribution. The variations of these quantities with respect to axial distance have been shown in Figs. 2-11. In Figs. 2-6, effect of porosity is shown graphically. In these figures, solid line corresponds to thermal double porous material (TDP) and small dashes line corresponds to thermal single porous material (TSP). Also, the effect of relaxation time is depicted graphically in Figs. 7-11. In Figs. 7-11, where solid line corresponds to Lord-Shulman (LS) theory of thermo elasticity and small dashes line corresponds to coupled (CT) theory of thermo elasticity.

9.1 Effect of porosity

Fig. 2 shows that the value of lateral deflection w initially increases for $1\le x\le 3.8$ and decreases onwards with the increase in the value of axial distance x. Similar behavior of variation is shown for both TDP and TSP but the magnitude of values is more for TDP in comparison to that of TSP. From Fig. 3, it is evident that for TDP, the value of axial stress t_x increases for $1\le x\le 2$, decreases for $2<x\le 9$ and again increases afterwards. Although, the trend and behavior of variation is similar for both TDP and TSP with difference in the magnitude of values. The values are higher for TSP than that of the values for TSP except for the region $x>9$ where the trend gets reversed. Fig. 4 depicts that the values of volume fraction field φ decreases initially and then increases slowly and steadily as $x > 2$. It is also found that due to effect of porosity, the magnitude of values are more for TDP in comparison to the values for TSP. Fig. 5 represents that the value of temperature distribution T increases monotonically with increase in axial distance x for both TDP and TSP but the magnitude of values of T are higher in case of TDP as compared to the values for TSP due to the effect of porosity. Fig. 6 shows that variation of axial displacement u shows similar pattern for both TDP and TSP near the point of application of source whereas the behavior of variation is opposite as moving away from the source. The magnitude of values of u is more near the application point of the source and it decreases as moving away from the source in case of TDP while for TSP, the value initially increases and becomes oscillatory as x increases.

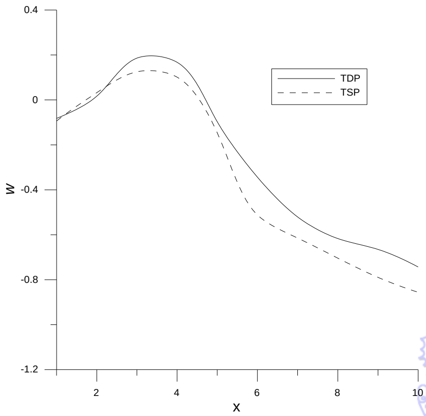

Fig. 2 Variation of lateral deflection w against axial distance x

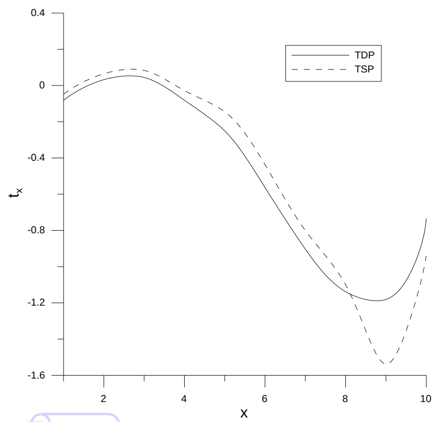

Fig. 3 Variation of axial stress t_x against axial distance x

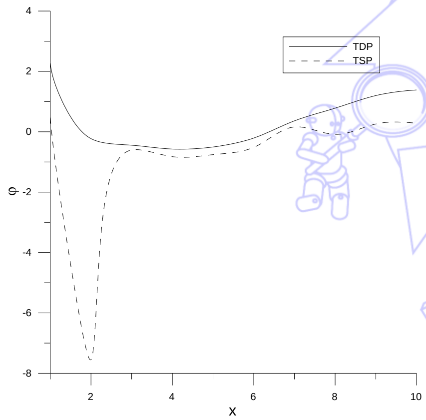

Fig. 4 Variation of volume fraction field φ against axial distance x

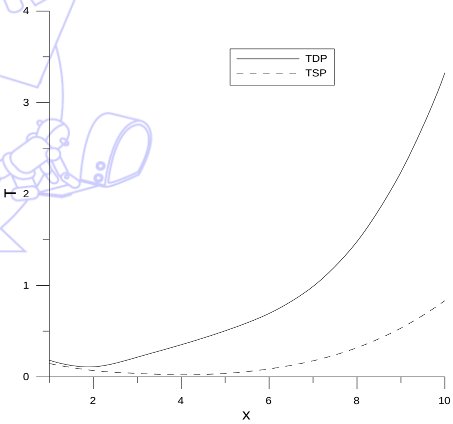

Fig. 5 Variation of temperature distribution T against axial distance x

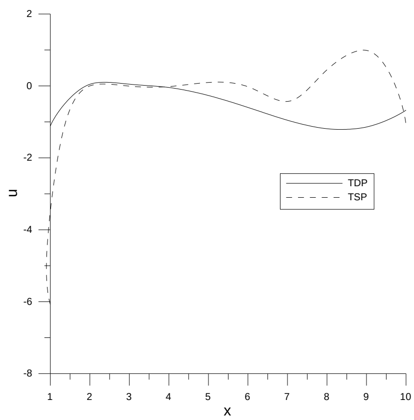

Fig. 6 Variation of axial displacement u against axial distance x

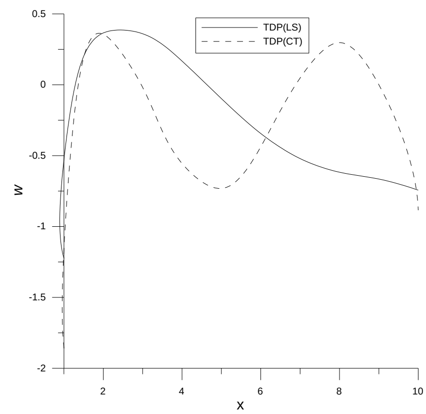

Fig. 7 Variation of lateral deflection w against axial distance x

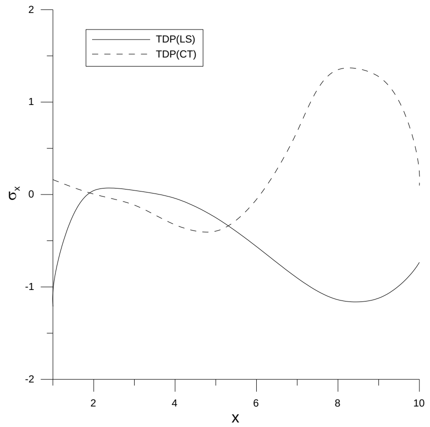

Fig. 8 Variation of axial stress t_x against axial distance x

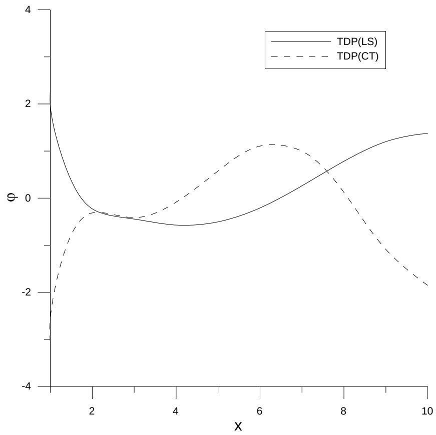

Fig. 9 Variation of volume fraction field φ against axial distance x

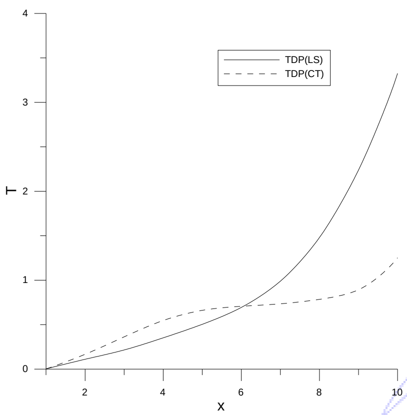

Fig. 10 Variation of temperature distribution T against axial distance x

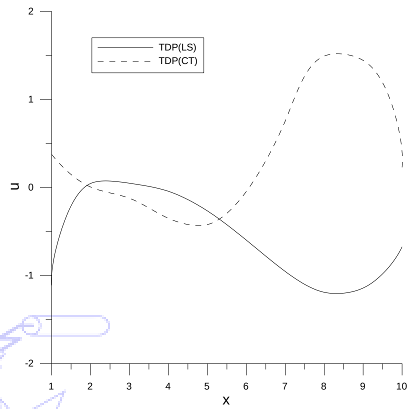

Fig. 11 Variation of axial displacement u against axial distance x

9.2 *Effect of thermal relaxation time*

Fig. 7 depicts that for LS theory, the value of lateral deflection w increases for $1 \le x \le 2.6$ and decreases afterwards with the increase in the value of axial distance x while for CT theory , it increases for the regions $1 \le x \le 2$, $5 \le x \le 8$ and decreases in the subsequent regions . From Fig. 8, it is noticed that due to effect of relaxation time parameter, the magnitude of values of axial stress is more for CT theory of thermo elasticity at both, near and away from the source application point. The trend and behavior of variation is opposite for LS and CT theories of thermo elasticity. Fig. 9 shows that the value of volume fraction field φ decreases initially and then increases as $x > 2$. It is also found that due to relaxation time effect, the value decreases near the application point of the source for LS theory while an opposite behavior is noticed in case of CT theory of thermo elasticity. Fig. 10 represents that for LS theory, the value of temperature distribution T increases monotonically with increase in axial distance x while for CT theory, it increases slowly and steadily as x increases. The magnitude of values is higher for CT theory near the source application point while an opposite trend is noticed away from the source. From Fig. 11, it is evident that for LS theory, the value of axial displacement u increases for $1 \le x \le 2$, decreases for $2 < x \le 8$ and increases afterwards. Due to relaxation time effect, the magnitude of values of u is more for CT theory in comparison to LS theory except for the region $2 \le x \le 5.3$ where an opposite trend of variation is noticed.

10. Conclusions

In this work, vibration analysis of a thermo elasticmicro beam with double porosity structure in context of Lord-Shulman theory of thermo elasticity, subjected to sinusoidal pulse heating is studied. Effects of porosity and thermal relaxation time parameter are shown graphically on lateral deflection, axial stress, axial displacement, volume fraction field and temperature distribution. It is observed that porosity has a significant effect on the all the physical quantities. It has both increasing as well as decreasing effect on the resulting quantities. Also, all the field quantities are observed to be very sensitive towards the thermal relaxation time parameter which shows that it is very important to take into account the relaxation time parameter.

This type of study is useful due to its physical application in geophysics, rock mechanics, mechanical engineering, civil engineering and industrial sectors. The results obtained in this investigation should prove to be beneficial for the researchers working on the theory of thermo elasticity with double porosity structure. The introduction of double porous parameter to the thermo elastic medium represents a more realistic model for further studies.

References

[1] H. Lord and Y. Shulman, "A generalized dynamical theory of thermo elasticity," Journal of Mechanics and Physics of Solids, vol. 15, pp. 299-309, 1967.

[2] M. A. Biot, "General theory of three-dimensional consolidation," Journal of Applied Physics, vol. 12, pp. 155-164, 1941.

[3] G. I. Barenblatt, I. P. Zheltov, and I. N. Kochina, "Basic concept in the theory of seepage of homogeneous liquids in fissured rocks (strata)," Journal of Applied Mathematics and Mechanics, vol. 24, pp. 1286-1303, 1960.

[4] E. C. Aifantis, "Introducing a multi-porous medium," Developments in Mechanics, vol. 8, pp. 209-211, 1977.

[5] E. C. Aifantis, "On the response of fissured rocks," Developments in Mechanics, vol. 10, pp. 249-253, 1979.

[6] E. C. Aifantis, "On the problem of diffusion in solids,"Acta Mechanica, vol. 37, pp. 265-296, 1980.

[7] R. K. Wilson and E .C. Aifantis, "On the theory of consolidation with double porosity," International Journal of Engineering Science, vol. 20, no. 9, pp. 1009-1035, 1984.

[8] M. Y. Khaled, D. E. Beskos and E. C. Aifantis, "On the theory of consolidation with double porosity-III," International Journal of Numerical and Analytical Methods in Geomechanics, vol. 8, pp. 101-123, 1984.

[9] D. E. Beskos and E. C. Aifantis, "On the theory of consolidation with double porosity-II," International Journal of Engineering Science, vol. 24, pp. 1697-1716, 1986.

[10] N. Khalili and A. P. S. Selvadurai, "A fully coupled constitutive model for thermo-hydro –mechanical analysis in elastic media with double porosity," Geophysical Research Letters, vol. 30, pp. 2268-2271, 2003.

[11] M. Svanadze, "Fundamental solution in the theory of consolidation with double porosity," Journal of the Mechanical Behavior of Materials, vol. 16, pp. 123-130, 2005.

[12] M. Svanadze, "Plane waves and boundary value problems in the theory of elasticity for solids with double porosity," Acta Applicande Mathematicae, vol. 122, pp. 461-470, 2012.

[13] B. Straughan, "Stability and uniqueness in double porosity elasticity," International Journal of Engineering Science, vol. 65, pp. 1-8, 2013.

[14] J. W. Nunziato and S. C. Cowin, "A nonlinear theory of elastic materials with voids," Archives of Rational Mechanics and Analysis, vol. 72, pp. 175-201, 1979.

[15] S. C. Cowin and J .W. Nunziato, "Linear elastic materials with voids," Journal of Elasticity, vol. 13, pp. 125-147, 1983.

[16] D. Iesan and R. Quintanilla, "On a theory of thermo elastic materials with a double porosity structure," Journal of Thermal Stresses, vol. 37, pp. 1017-1036, 2014.

[17] A. Dimarogonas, Vibration for engineers. Prentice-Hall, Inc., 2nd edition, 1996.

[18] L. Meirovitch, Fundamentals of vibrations. McGraw-Hill, International Edition, 2001.

[19] B. A. Boley, "Approximate analyses of thermally induced vibrations of beams and plates," Journal of Applied Mechanics, vol. 39, pp. 212-216, 1972.

[20] G. D. Manolis and D. E. Beskos, "Thermally induced vibrations of beam structures," Computer Methods in Applied Mechanics and Engineering, vol. 21, pp. 337-355, 1980.

[21] N. S. Al-Huniti, M. A. Al-Nimr, and M. Naij, "Dynamic response of a rod due to a moving heat source under the hyperbolic heat conduction model," Journal of Sound and Vibration, vol. 242, pp. 629-640, 2001.

[22] B. Biondi and S. Caddemi, "Closed form solutions of Euler-Bernoulli beams with singularities," International Journal of Solids and Structures, vol. 42, pp. 3027-3044, 2005.

[23] D. N. Fang, Y. X. Sun, and A. K. Soh, "Analysis of frequency spectrum of laser-induced vibration of microbeam resonators," Chinese Physics Letters, vol. 23, pp. 1554-1557, 2006.

[24] J. N. Sharma, and D. Grover, "Thermo elastic vibrations in micro-/nano-scale beam resonators with voids," Journal of Sound and Vibration, vol. 330, pp. 2964-2977, 2011.

[25] I. Esen, "A new FEM procedure for transverse and longitudinal vibration analysis of thin rectangular plates subjected to a variable velocity moving load along an arbitrary trajectory," Latin American Journal of Solids and Structures, vol. 12, pp. 808-830, 2015.

[26] A. M. Dehrouyeh-Semnani, M. Dehrouyeh, M. Torabi-Kafshgari, and M. Nikkah-Bahrami, "A damped sandwich beam model based on symmetric-deviatoriccouple stress theory," International Journal of Engineering Science, vol. 92, pp. 83-94, 2015.

[27] A. M. Dehrouyeh-Semnani, M. Dehrouyeh, M. Torabi-Kafshgari, and M. Nikkah-Bahrami, "Free flexural vibration of geometrically imperfect functionally graded microbeams," International Journal of Engineering Science, vol. 105, pp. 56-79, 2016.

[28] M. Mojahedi, and M. Rahaeifard, "A size-dependent model for coupled 3D deformations of nonlinear microbridges," International Journal of Engineering Science, vol. 100, pp. 171-182, 2016.

[29] G. Honig, and U. Hirdes, "A method for the numerical inversion of the Laplace transforms," Journal of Computational and Applied Mathematics, vol. 10, pp. 113-132, 1984.

[30] D. Tzou, Macro-to-Micro Heat transfer, Taylor& Francis, Washington DC, 1996.

[31] H. Sherief, and H. Saleh, "A half space problem in the theory of generalized thermo elastic diffusion," International Journal of Solids and Structures. vol. 42, pp. 4484-4493, 2005.

[32] N. Khalili, "Coupling effects in double porosity media with deformable matrix," Geophysical Research Letters, vol. 30, no. 22, pp. 2153, doi: 10.1029/2003GL018544, 2003.

[33] A. Altinörs, and Halilönder, "A double-porosity model for a fractured aquifer with non-darcian flow in fractures," Hydrological Sciences-Journal-des Sciences Hydrologiques, vol. 53, no. 4, pp. 868-882, 2008.

[34] A. R. Bagherieh, N. Khalili, G. Habibagahi, and A. Ghahramani, "Drying response and effective stress in a double porosity aggregated soil," Engineering Geology, vol. 105, pp. 44-50, 2009.

[35] T. D. Tran Ngo, J. Lewandowska and H. Bertin, "Experimental evidence of the double-porosity effects in geomaterials,"Acta Geophysica, vol. 62, no. 3, pp. 642-655, 2014.

Appendix I

$$a_{26} = s(1+\tau_0 s),\ a_{27} = -s(1+\tau_0 s)a_{22},\ a_{28} = -s(1+\tau_0 s)a_{23},\ a_{29} = -s(1+\tau_0 s)a_{24},\ a_{30} = -\left\{a_{21} + s(1+\tau_0 s)a_{25}\right\},$$
$$n_1 = -\left(a_6 + a_{10} + s^2\right),\ n_2 = -\left(a_8 + a_{11}\right),\ n_3 = -\left(a_{18} + a_{14}\right),\ n_4 = -\left(a_{16} + a_{19} + s^2\right),$$
$$r_1 = a_5 a_{15} - a_7 a_{13},\ r_2 = a_5(a_{15}a_{30} + n_4) - a_{13}n_2 + n_1 a_{15} - a_7(a_{13}a_{30} + n_3),$$
$$r_3 = n_1(a_{15}a_{30} + n_4) + a_5(n_4 a_{30} - a_{20}a_{29}) - a_7(n_3 a_{30} - a_{20}a_{28}) - n_2(a_{13}a_{30} + n_3) + a_{12}(a_{13}a_{29} - a_{15}a_{28}),$$
$$r_4 = n_1(n_4 a_{30} - a_{20}a_{29}) + a_{12}(n_3 a_{29} - n_4 a_{28}) + n_2(a_{20}a_{28} - n_3 a_{30}),\ r_5 = a_9 a_{15} - a_7 a_{17},$$
$$r_6 = a_9(a_{15}a_{30} + n_4) - a_7(a_{17}a_{30} - a_{20}a_{27}) - n_2 a_{17} - a_{12}a_{15}a_{27},$$
$$r_7 = a_9(n_4 a_{30} - a_{20}a_{29}) + a_{12}(a_{17}a_{29} - n_4 a_{27}) - n_2(a_{17}a_{30} - a_{20}a_{27}),$$

$$r_8 = a_9 a_{13} - a_5 a_{17},\ r_9 = a_9(a_{13}a_{30} + n_3) - n_1 a_{17} - a_5(a_{17}a_{30} - a_{20}a_{27}) - a_{12}a_{10}a_{27},$$
$$r_{10} = a_9(n_3 a_{30} - a_{20}a_{28}) - n_1(a_{30}a_{17} - a_{27}a_{20}) + a_{12}\left(a_{17}a_{28} - n_3 a_{27}\right),$$
$$r_{11} = a_{27}(a_5 a_{12} + a_7 a_{13}),\ r_{12} = a_9(a_{13}a_{29} - a_{15}a_{28}) + a_5(n_4 a_{27} - a_{17}a_{29}) + a_{27}(n_1 a_{15} + g_3 a_{13})$$
$$-a_7(a_{17}a_{28} - n_3 a_{27}),\ r_{13} = a_9(n_3 a_{29} - n_4 a_{28}) - n_1(a_{17}a_{29} - n_4 a_{27}) - n_2(a_{17}a_{28} - n_3 a_{27}),$$

$$B_1 = \left(r_2 + a_2 r_5 - a_3 r_8 - a_4 r_{11}\right)/r_1,\ B_2 = \left(a_1 r_1 s^2 + a_2 r_6 - a_3 r_9 - a_4 r_{12} + r_3\right)/r_1,$$
$$B_3 = \left(a_1 r_3 s^2 + a_2 r_7 - a_3 r_{10} - a_4 r_{13} + r_4\right)/r_1,\ B_4 = \left(a_1 r_3 s^2\right)/r_1,\ B_5 = \left(a_1 r_4 s^2\right)/r_1$$

Appendix II

$$g_{1i} = -\left\{r_5\lambda_i^6 + r_6\lambda_i^4 + r_7\lambda_i^2\right\}/\left\{r_1\lambda_i^6 + r_2\lambda_i^4 + r_3\lambda_i^2 + r_4\right\},\ g_{2i} = \left\{r_8\lambda_i^6 + r_9\lambda_i^4 + r_{10}\lambda_i^2\right\}/\left\{r_1\lambda_i^6 + r_2\lambda_i^4 + r_3\lambda_i^2 + r_4\right\},$$
$$g_{3i} = -\left\{r_{11}\lambda_i^6 + r_{12}\lambda_i^4 + r_{13}\lambda_i^2\right\}/\left\{r_1\lambda_i^6 + r_2\lambda_i^4 + r_3\lambda_i^2 + r_4\right\};\ i = 1,\ 2,\ 3, \ldots,\ 5$$